German-Japanese Workshop 2024, Mainz, Germany, Sep. 25, 2024

Lee-Yang-zero ratio for critical-point searches

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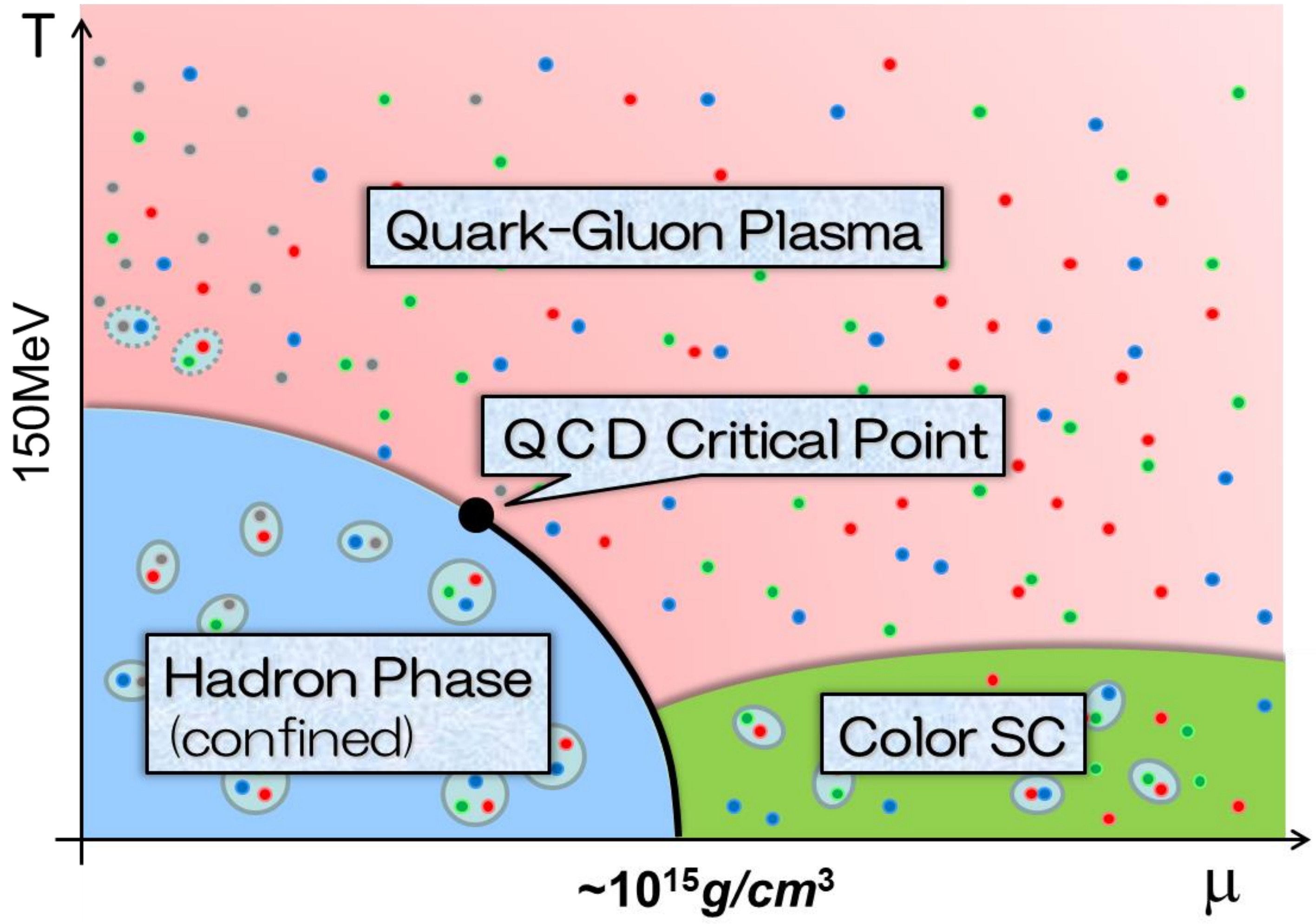
In collab. with Tatsuya Wada, Kazuyuki Kanaya
Special thanks to S. Ejiri

QCD Phase Diagram



Rich phase structure in QCD

- QCD critical point(s)
- color superconductivity

Sign problem

- difficulty in lattice QCD Monte-Carlo simulations at $\mu \neq 0$

Various approaches

- Taylor expansion method
- Imaginary chem. pot.
- Complex Langevin
- Lifschetz thimble
- Lee-Yang edge singularity
- …

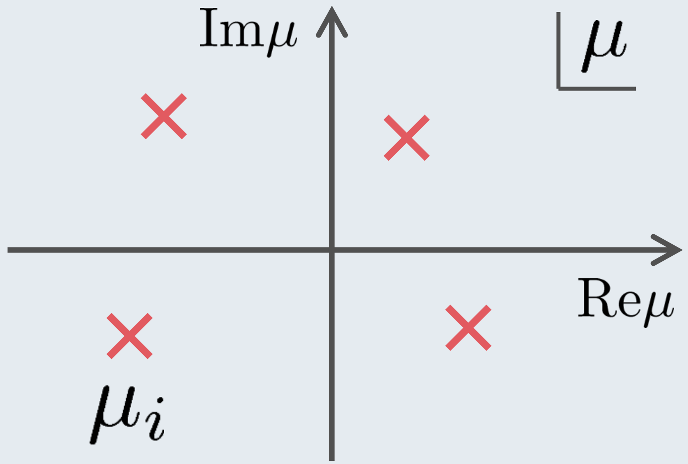
Lee-Yang Zeros

Yang, Lee; Lee, Yang ('52)

Partition Function $Z(T,\mu)$

Finite V ➤ Polynomial of μ (or T)

$$Z(T,\mu) = \prod_i (\mu - \mu_i)$$



➤ **zeros** on the complex plane

=Lee-Yang Zeros

Lee-Yang Zeros

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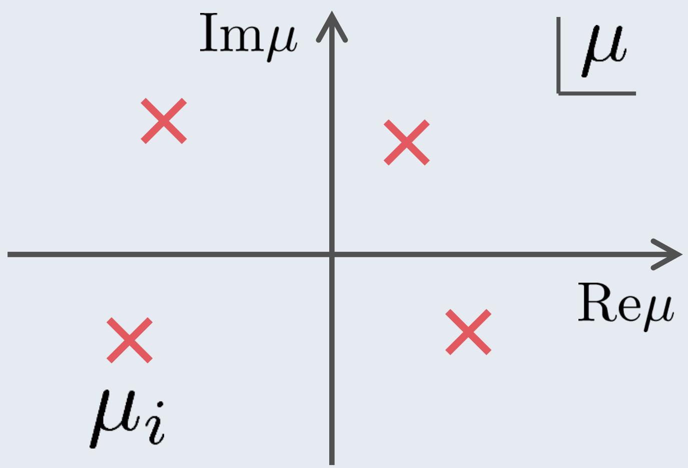
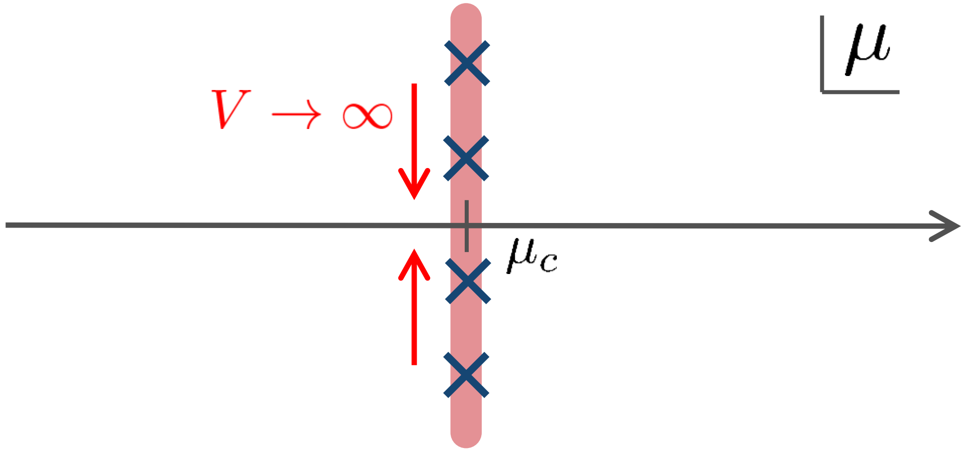


➤ **zeros** on the complex plane

=**Lee-Yang Zeros**

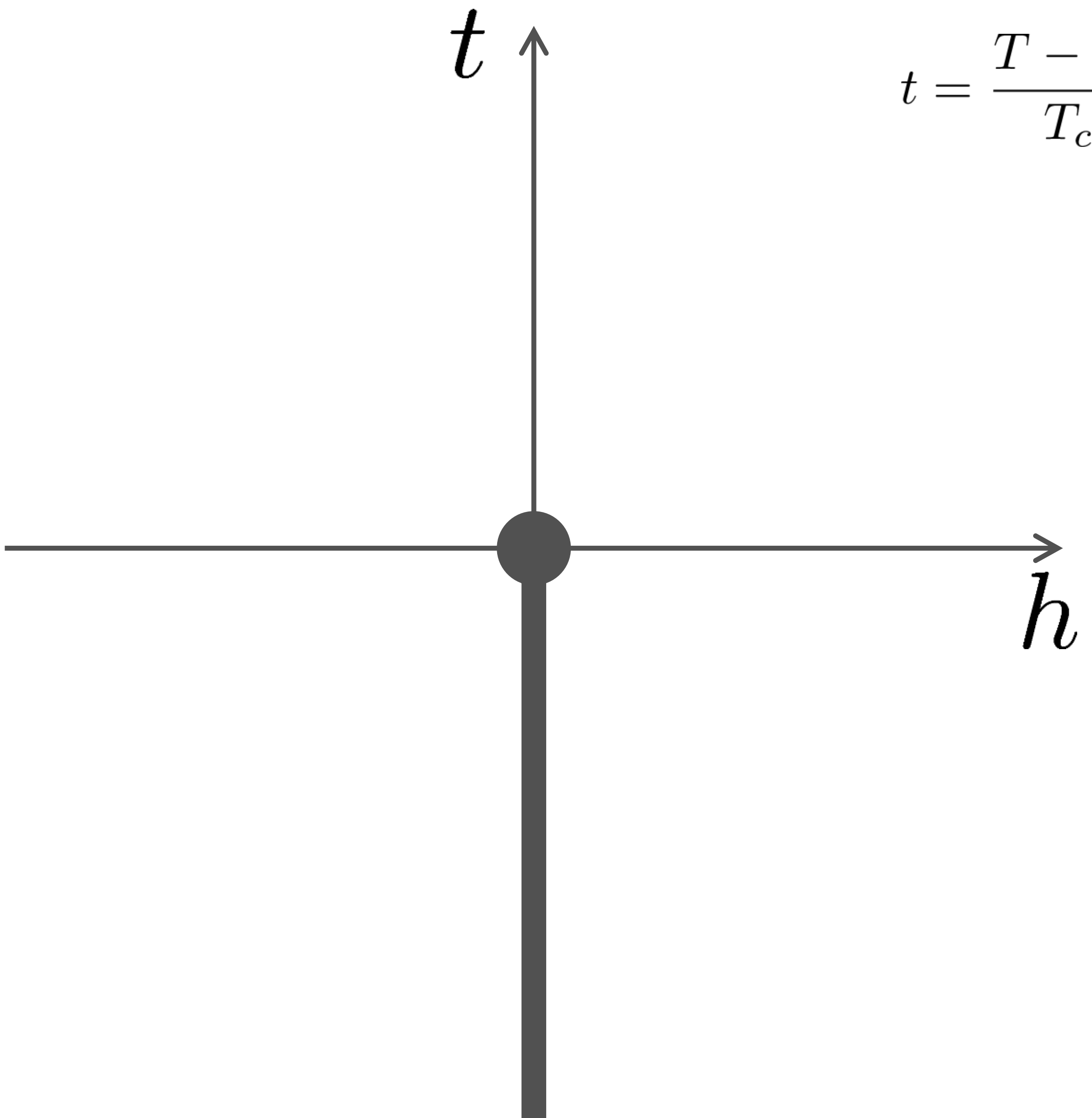
Phase Transition & LYZ

First-order transition
at $\mu = \mu_c$



— For $V \to \infty$, LYZs are accumulated on the line crossing the real axis at $\mu = \mu_c$.

LYZ around a Critical Point in Ising Model



$$t = \frac{T - T_c}{T_c}$$

1st-transition

singularity on the real h axis

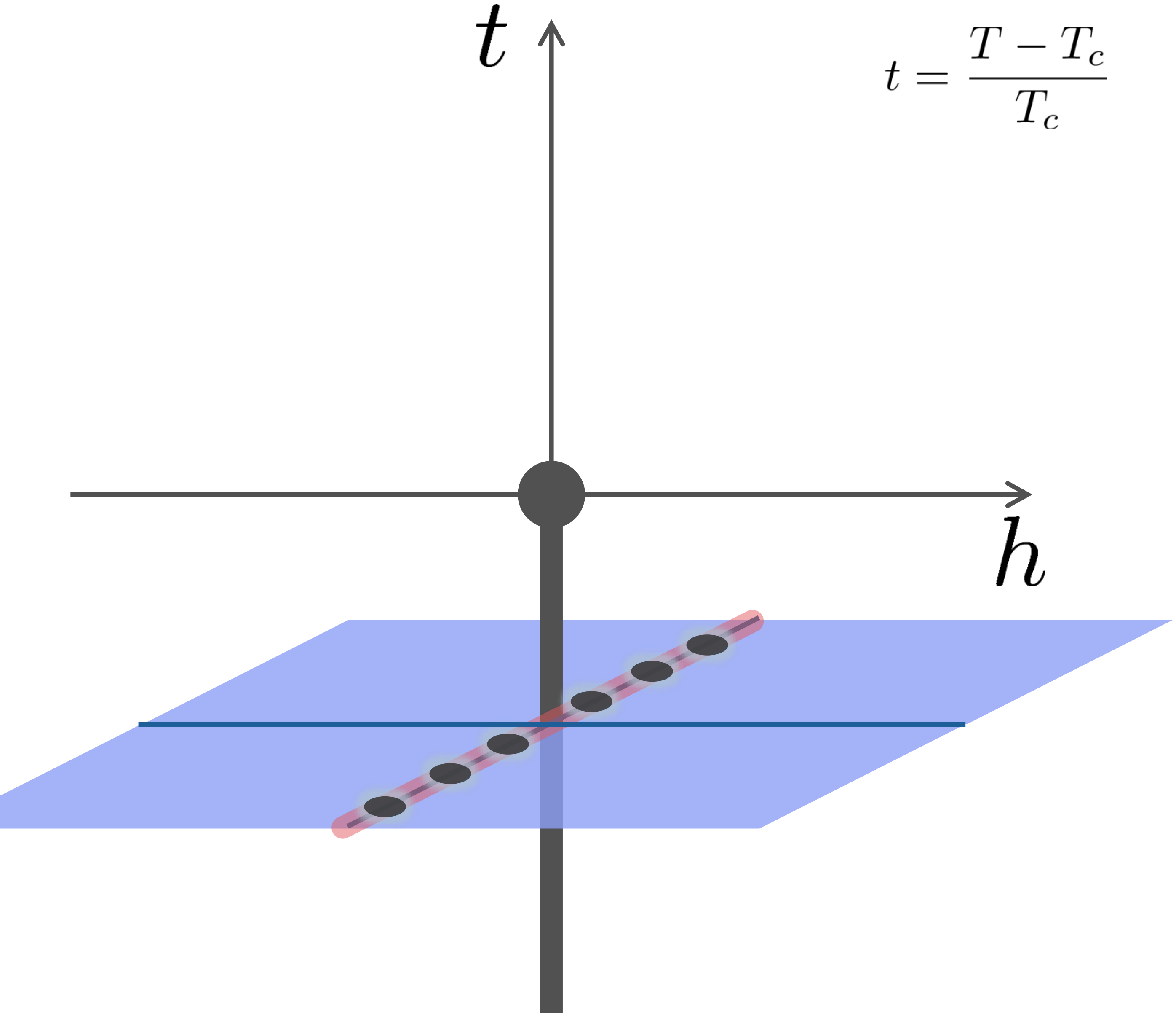
Crossover

no singularity on the real axis

Note:
LYZ in complex-h plane are purely imaginary.

Lee-Yang, 1952

LYZ around a Critical Point in Ising Model



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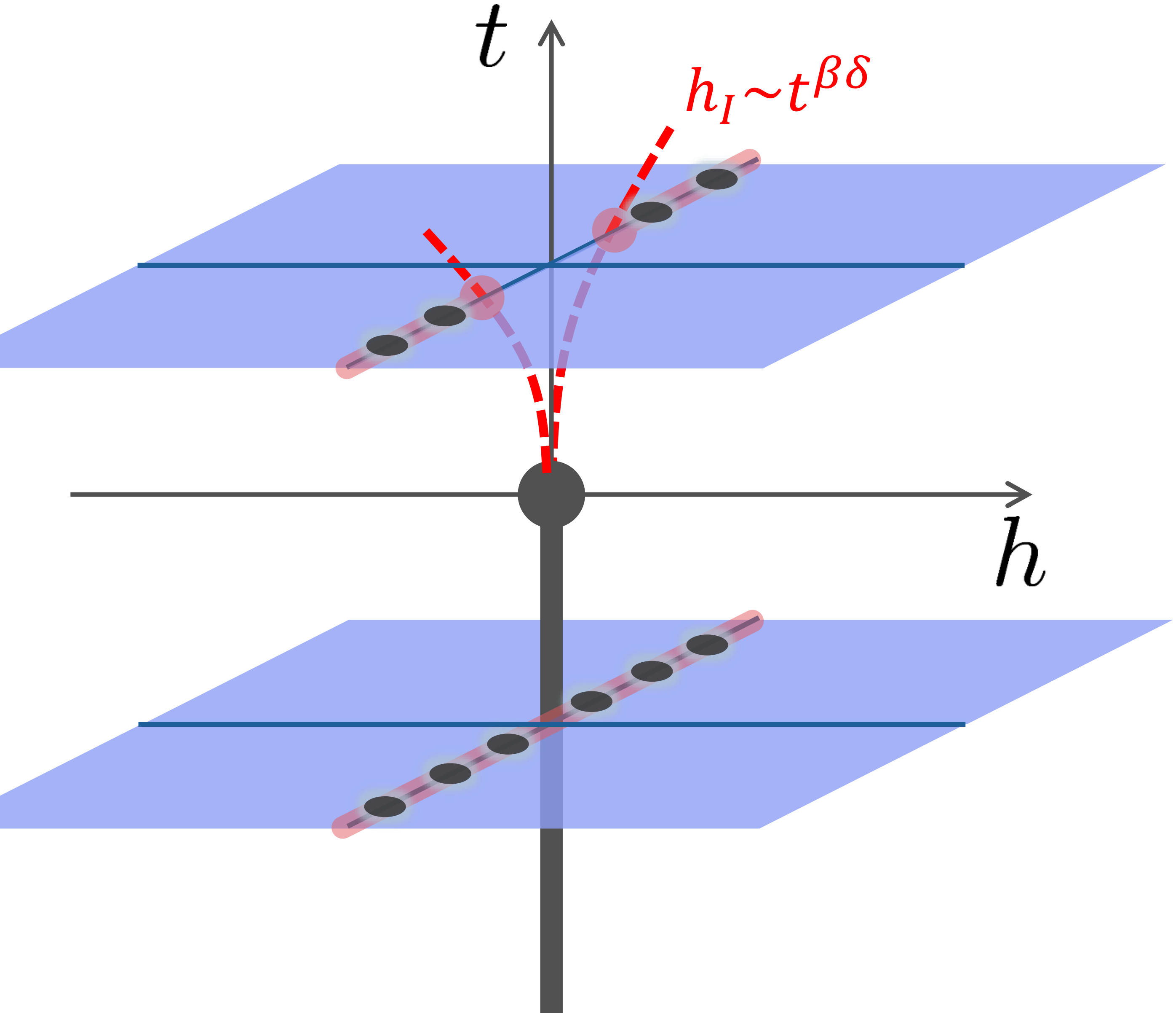


1st-transition

singularity on the real h axis

Crossover

no singularity on the real axis

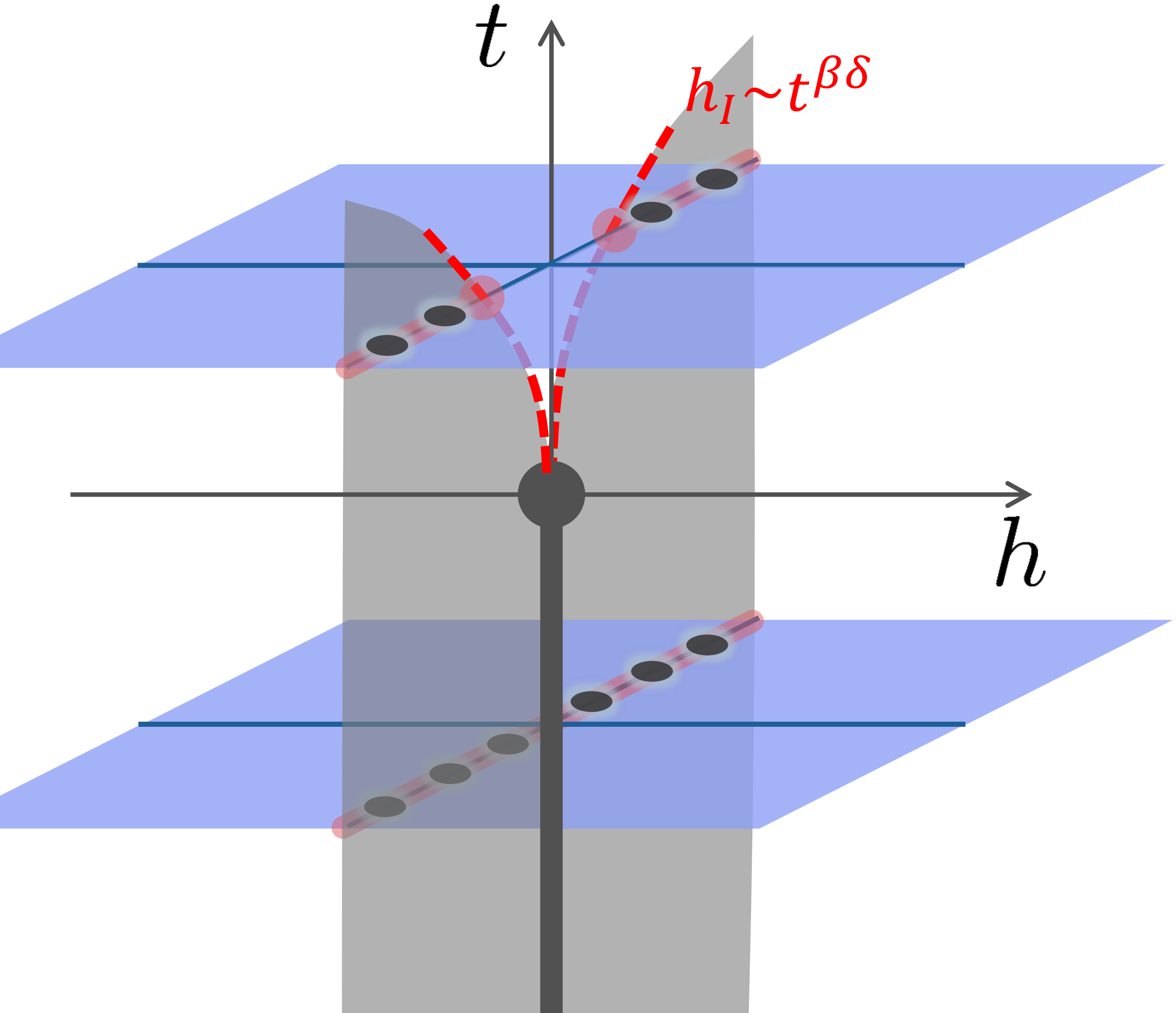
LY edge singularity

Starting from the CP

Its behavior is governed by the the scaling function.

$h_I \sim t^{\beta\delta}$

LYZ around a Critical Point in Ising Model



1st-transition

singularity on the real h axis

Crossover

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LY edge singularity

Starting from the CP

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Recent Progress in LYZ/LYES and Lattice

Analytic Structure

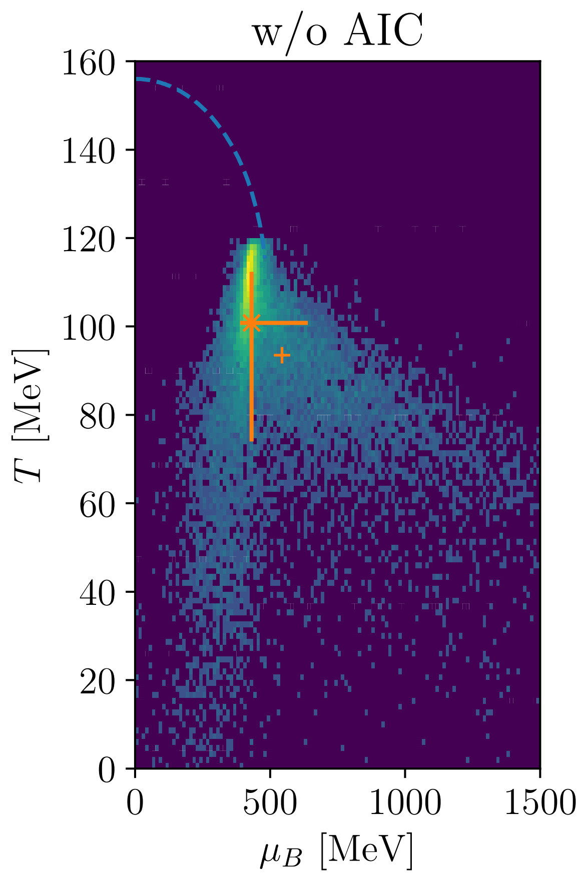
— Scaling functions, FRG, …

An, Mesterhazy, Stephanov ('16)
Johnson, Rennecke, Skokov ('23)
Karsch, Schmidt, Singh ('23)
…

Locating QCD-CP at $\mu \neq 0$ on the lattice?

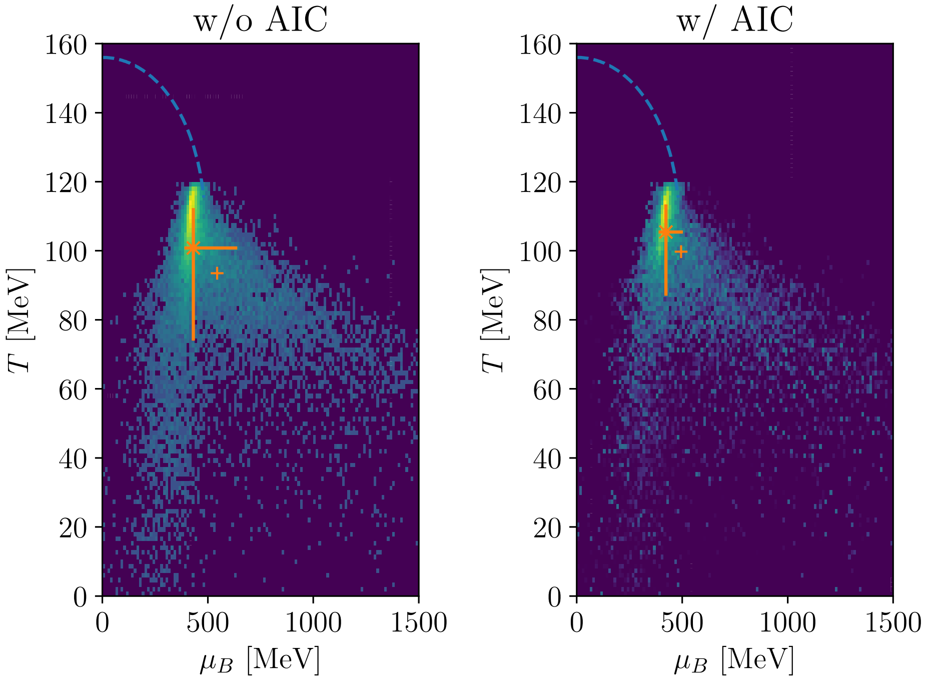
Clarke+, arXiv:2405.10196
Alexander+ Lattice2024

— Taylor exp. + Imaginary μ + Pade approx.
— Identify the 1st LYZ to be LYES



QCD-CP Search with Lee-Yang Zeros

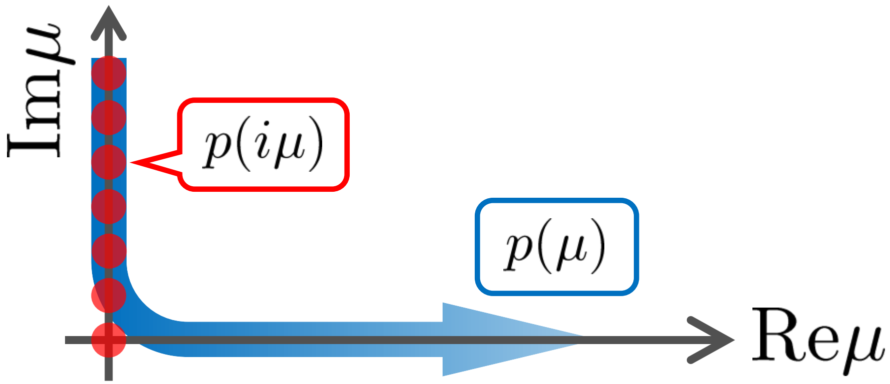
D. Clarke+ 2405.10196 [hep-lat]



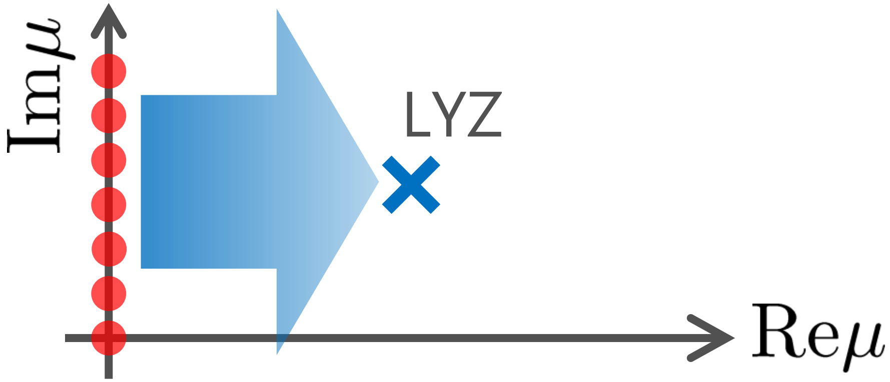
$$\begin{cases} \mu^{\mathrm{CEP}} = 422^{+80}_{-35}\mathrm{MeV} \\ T^{\mathrm{CEP}} = 105^{+8}_{-18}\mathrm{MeV} \end{cases}$$

See also, Alexander+ Lattice2024

Imaginary chem. pot.

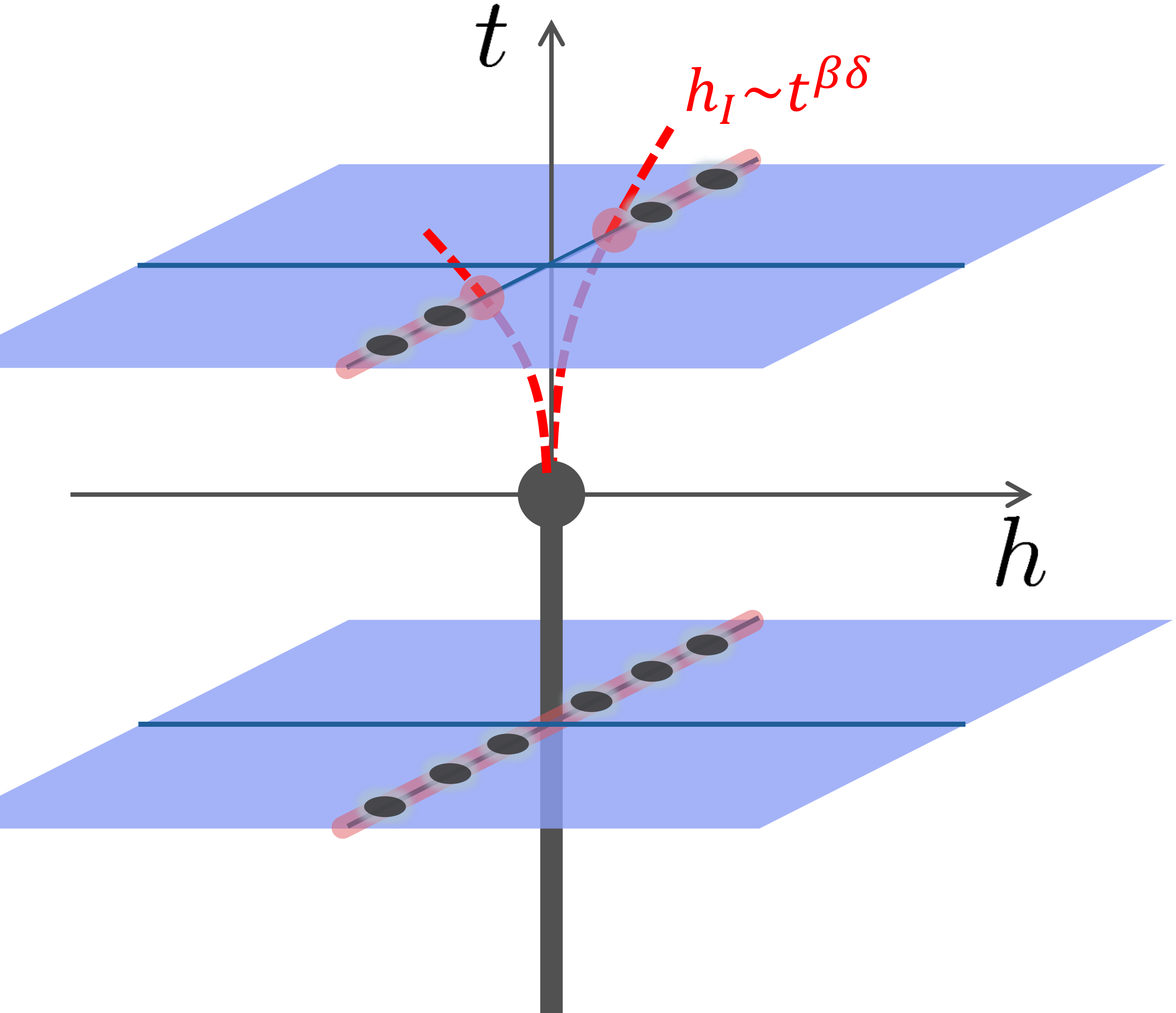


Lee-Yang zero



analytic continuation via Pade approx.

Purpose of This Study



On finite volume,

1st LYZ ≠ LYES

- Study finite-volume effects on the LYZ near CP.
- Propose a new method to explore the CP via LYZ.

Finite-Size Scaling

Scaling Hypothesis

$$F_{\text{sing}}(t, h, L^{-1}) = \tilde{F}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$Z_{\text{sing}}(t, h, L^{-1}) = \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h)$$

$$F = F_{\text{sing}} + F_{\text{reg}}$$

$$Z = Z_{\text{sing}} \times Z_{\text{reg}}$$

LYZ in the scaling region on finite volume

$$Z(t, h, L^{-1}) \sim \tilde{Z}_{\text{sing}}(L^{y_t} t, L^{y_h} h) = 0$$

$$\boxed{L^{y_h} h^{(i)} = \tilde{h}^{(i)}_{\text{LY}}(L^{y_t} t)}$$

LYZ in 3d-Ising Model

$$H = -\sum_{\langle i,j\rangle} s_i s_j - h\sum_i s_i$$

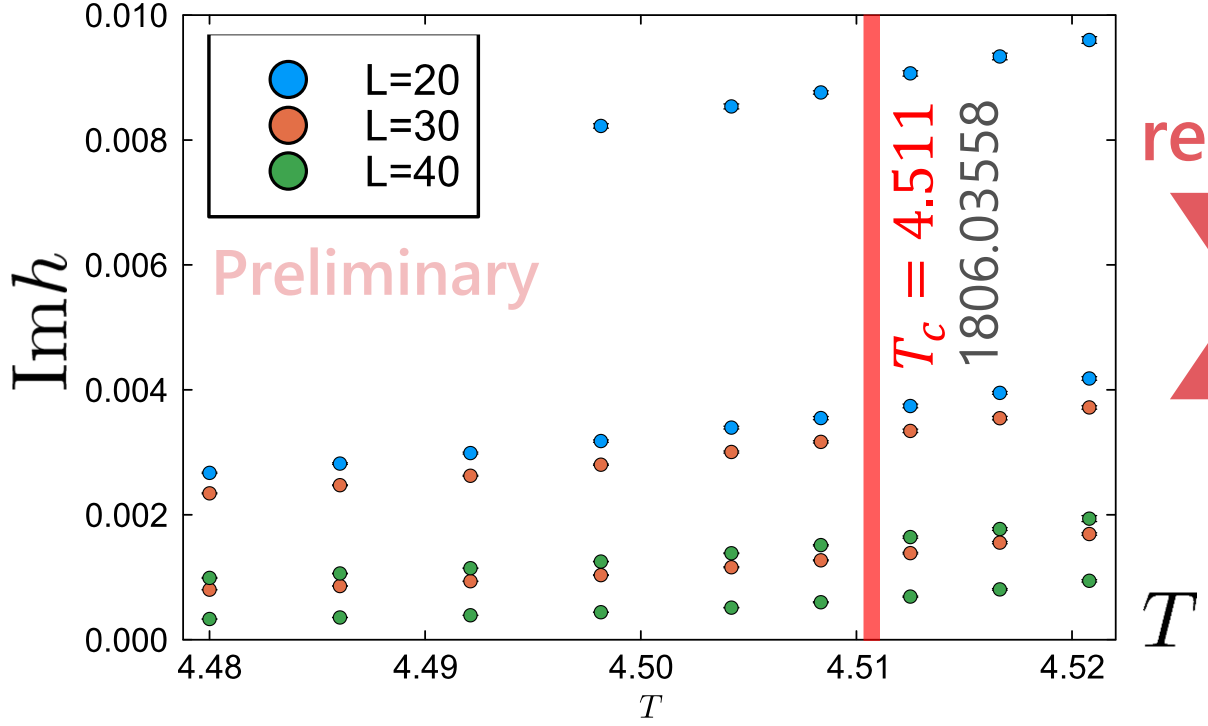
Monte-Carlo + reweighting

LYZ

$$\frac{Z(t,\tilde{h})}{Z(t,h)} = 0$$

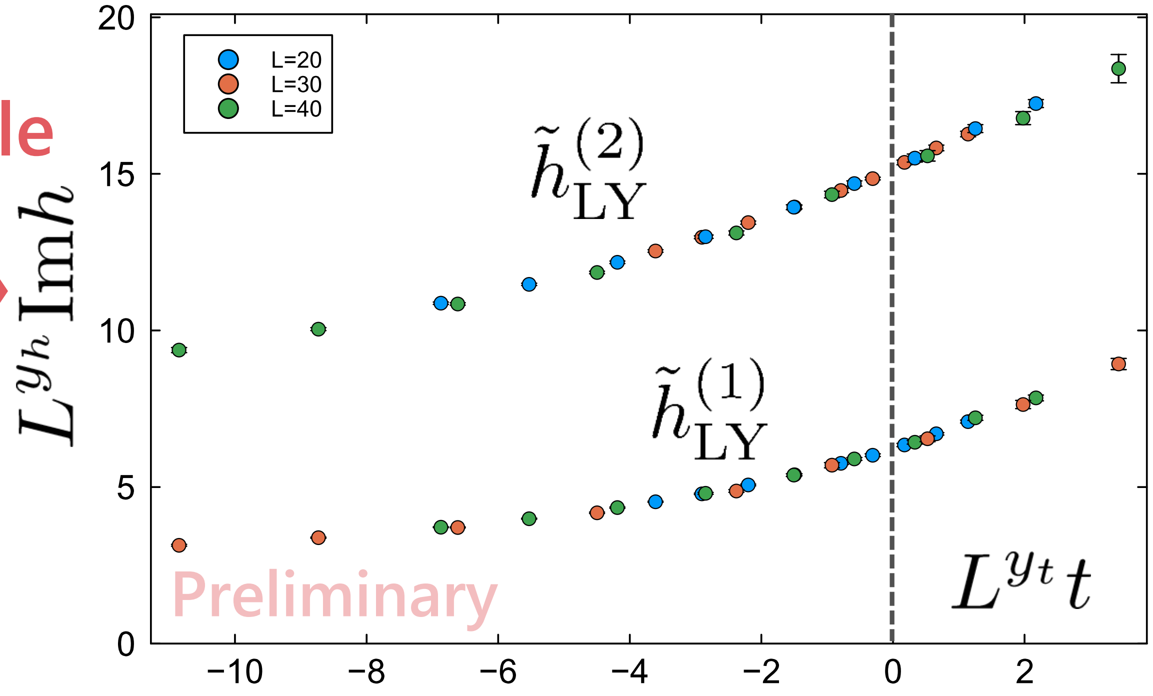
$$\boxed{L^{y_h} h^{(i)} = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)}$$

1st & 2nd LYZ



rescale

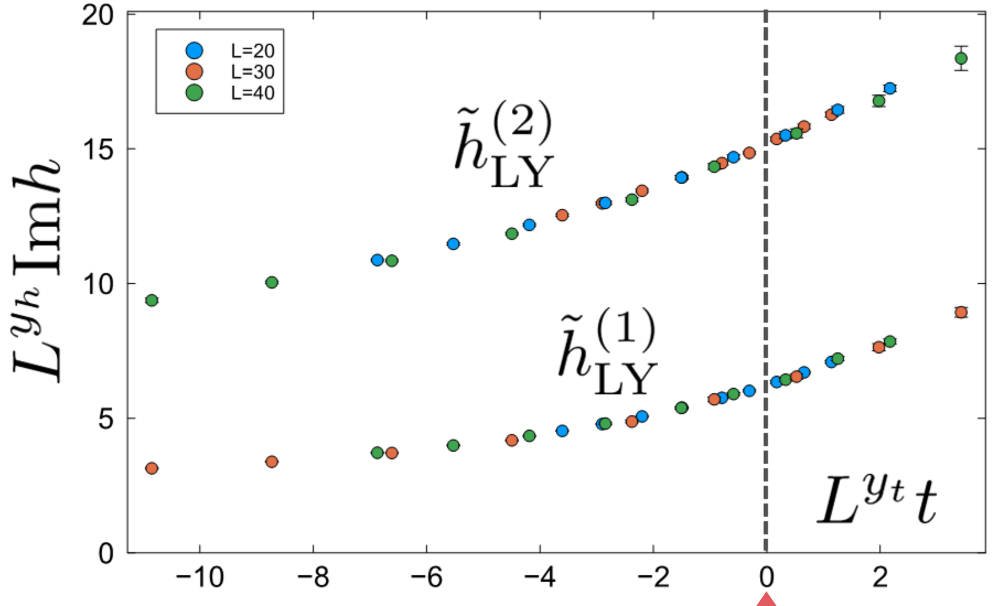
Rescaled



LYZ is away from the real axis at the CP on finite L.

Where is QCD Critical Point?

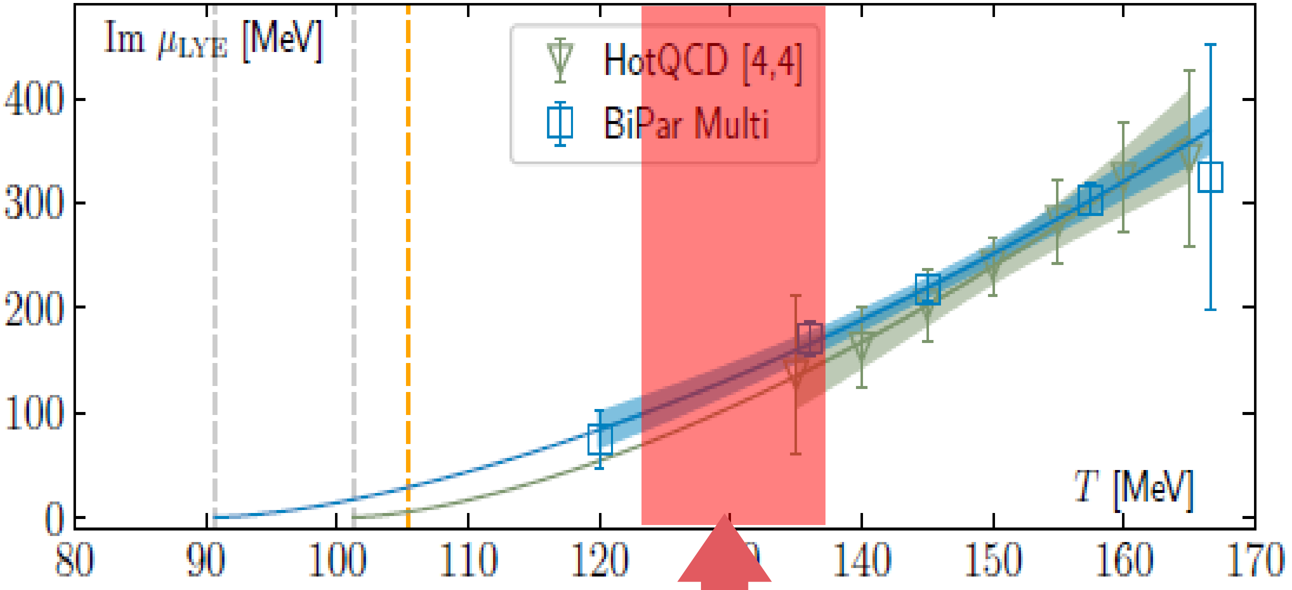
Ising model



$T = T_c$

LYZ in QCD

Clarke+, arXiv:2405.10196

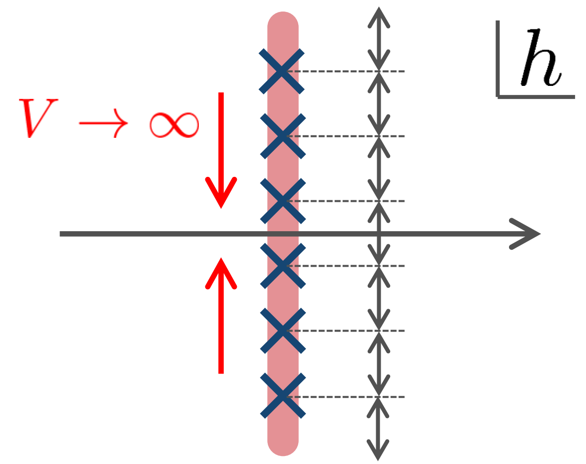


CP around here?

Lee-Yang Zero Ratio

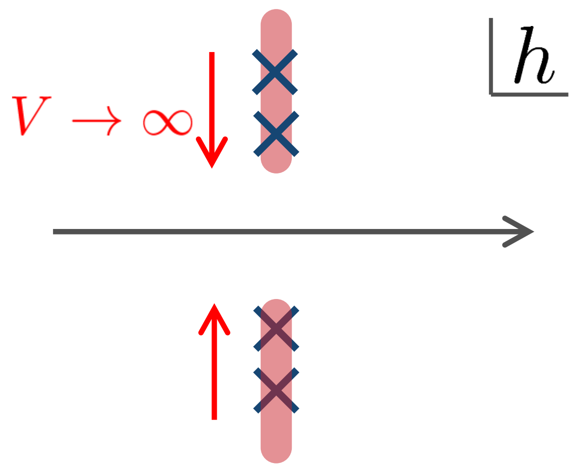
$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

First-Order Side $(t < 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} \frac{2n-1}{2m-1}$$

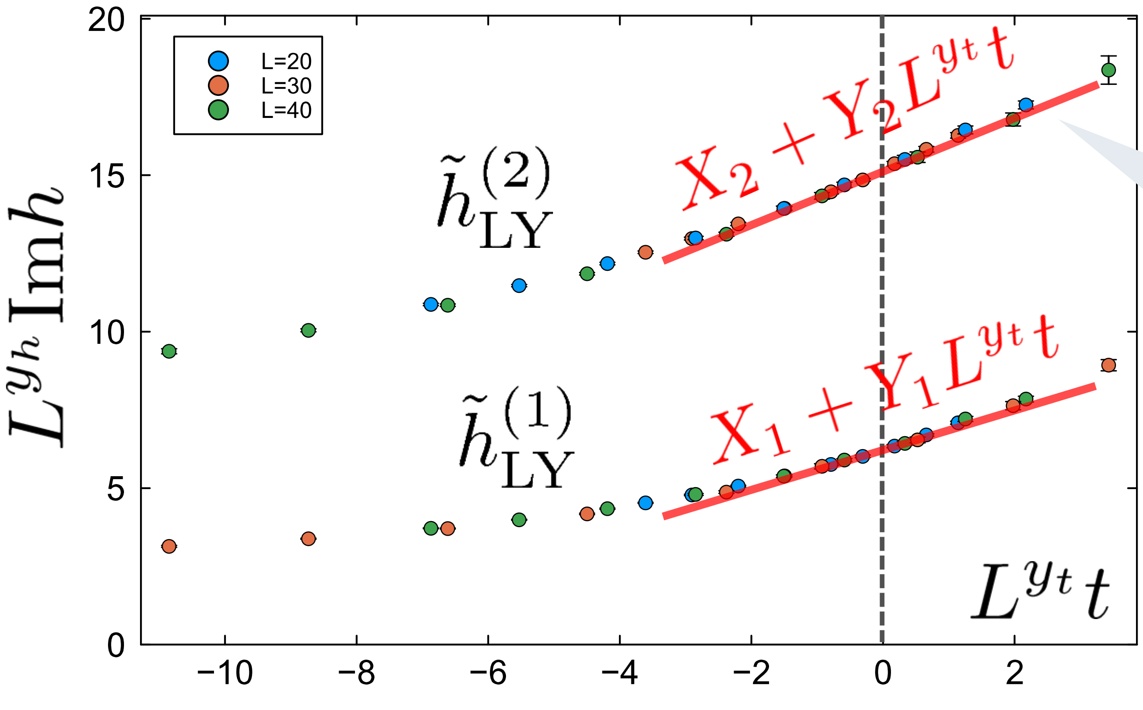
Crossover Side $(t > 0)$



$$R_{nm}(t) \xrightarrow[V\to\infty]{} 1$$

Lee-Yang Zero Ratio

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$



$$C_{nm} = \frac{Y_n}{X_n} - \frac{Y_m}{X_m}$$

Linear Approx. at t = 0

$$L^{y_h} h = \tilde{h}^{(i)}_{\mathrm{LY}}(L^{y_t} t)$$
$$= X_i + Y_i L^{y_t} t + \mathcal{O}(t^2)$$

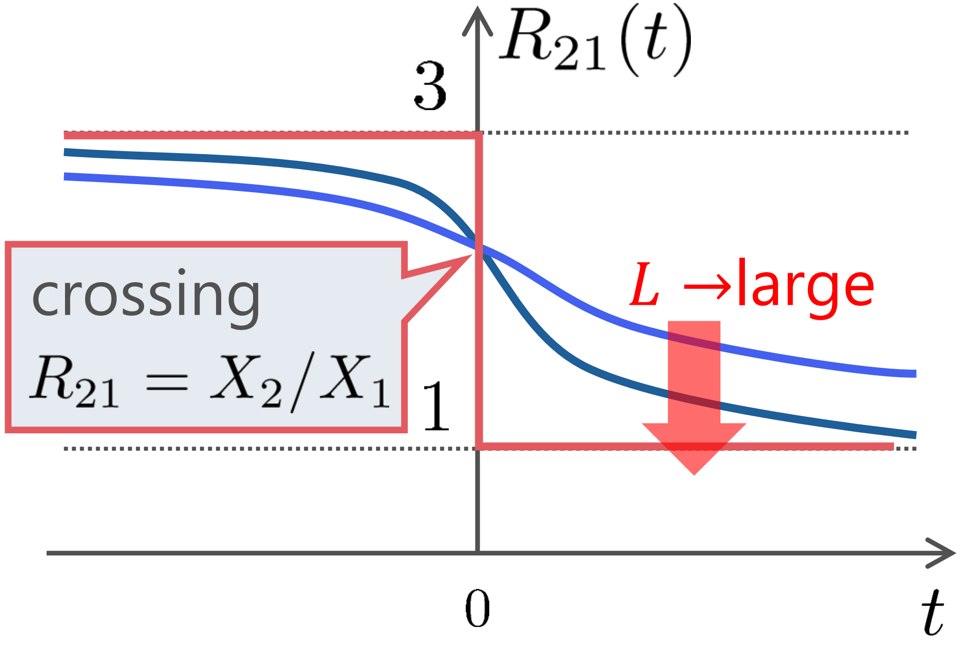
$$R_{nm}(t) = \frac{X_n}{X_m}\left(1 + C_{nm} t L^{y_t} + \mathcal{O}(t^2)\right)$$

LYZ Ratio $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{n1}(t) = \frac{X_n}{X_1}\left(1 + C_{n1} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$



— $R(0)$ is L independent, the universal value.

—Crossing point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

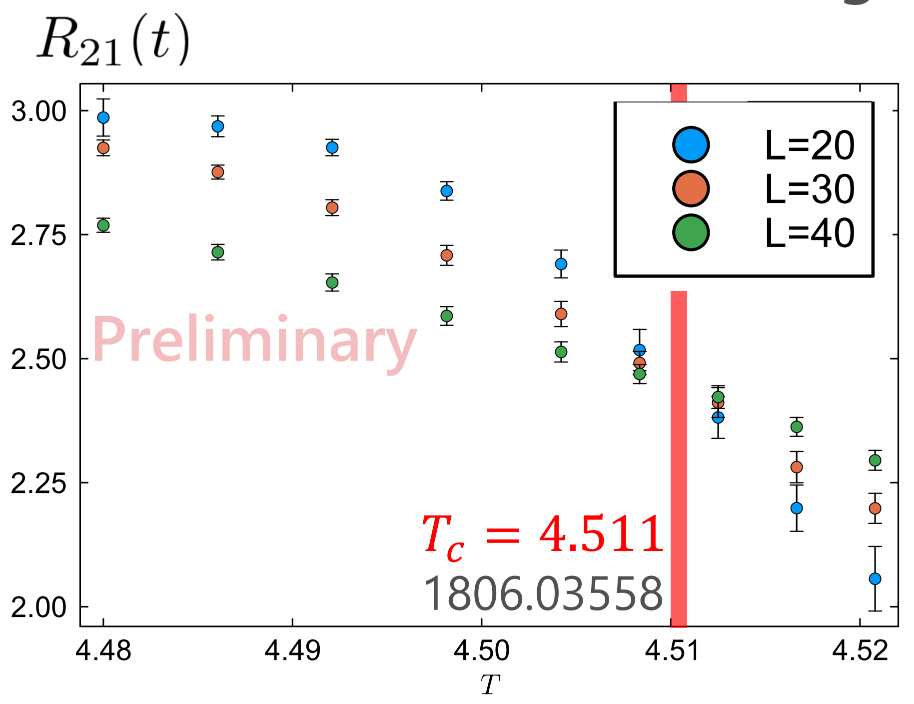
LYZ Ratio $R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$

$$R_{n1}(t) \xrightarrow[V\to\infty]{} \begin{cases} 2n-1 & t<0 \text{ (1st order)} \\ 1 & t>0 \text{ (crossover)} \end{cases}$$

$$R_{n1}(t) = \frac{X_n}{X_1}\left(1 + C_{n1} t L^{y_t} + \mathcal{O}(t^2)\right)$$

near $t = 0$

Numerical Result in 3d-Ising



$R_{21}(0) \simeq 2.42$

— $R(0)$ is L independent, the universal value.

—Crossing point of various L gives the CP.

—Reminiscent of Binder-cumulant analysis

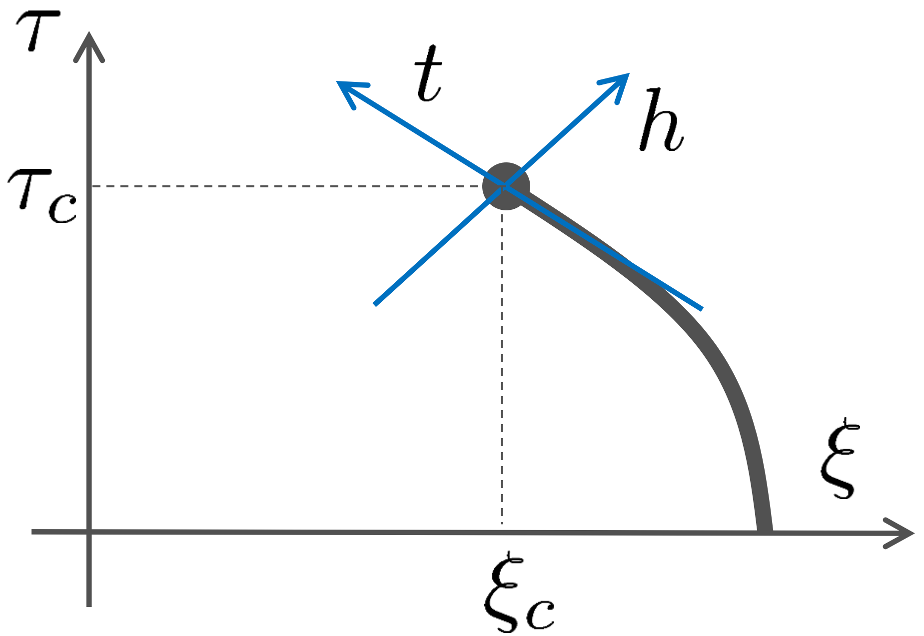
$$Y = y_t - y_h = -0.894$$

General CP

- CP on a $\tau - \xi$ plane
- LYZ on the complex ξ plane

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$



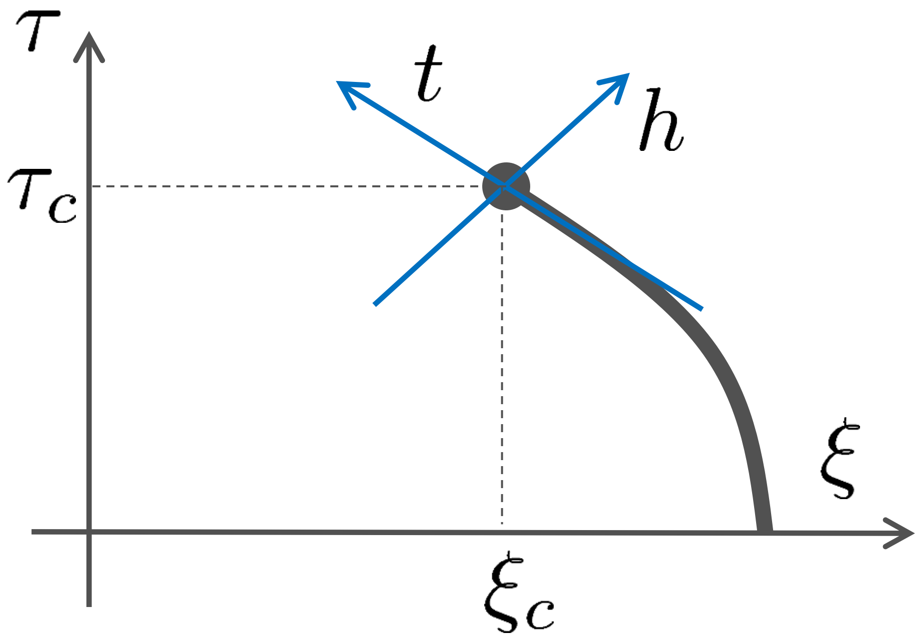
General CP

$\bar{y} = y_t - y_h = -0.894$

- CP on a $\tau - \xi$ plane
- **LYZ on the complex ξ plane**

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$

$$L^{y_h} h^{(n)} \simeq X_i + Y_i L^{y_t} t$$



$$\begin{cases} \xi_{\mathrm{R}}^{(n)} = \xi_c - \dfrac{a_{21}}{a_{22}} \delta\tau + \mathcal{O}(L^{2\bar{y}}) \\ \xi_{\mathrm{I}}^{(n)} = \dfrac{X_n}{a_{22}} L^{-y_h} + \dfrac{\det AY_n}{a_{22}^2} \delta\tau L^{\bar{y}} + \mathcal{O}(L^{2\bar{y}}) \end{cases}$$

$L \to \infty$

generalization

LY Edge Singularity

$$\begin{cases} \mathrm{Re}\xi_{\mathrm{LYES}} \simeq c_1 \tau \\ \mathrm{Im}\xi_{\mathrm{LYES}} \simeq c_2 \tau^{\beta\delta} \end{cases}$$

Stephanov, 2006

$\bar{y} = y_t - y_h = -0.894$

LYZ Ratio for General CP

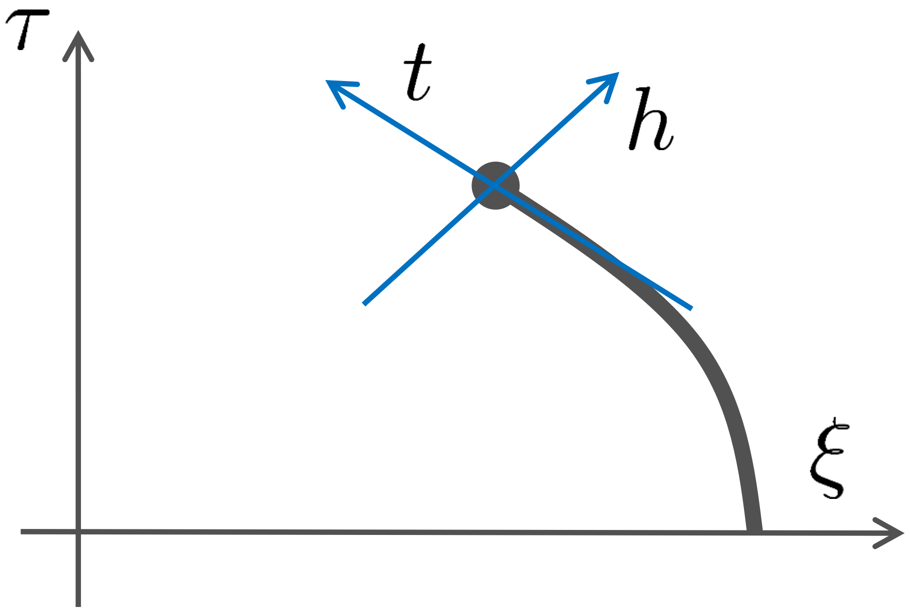
LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

$$C = \frac{\det A}{a_{22}}\left(\frac{Y_2}{X_2} - \frac{Y_1}{X_1}\right), \quad D = \frac{a_{12}^2}{a_{22}^2}(Y_1^2 - Y_2^2)$$

$$\begin{pmatrix} t \\ h \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} \tau - \tau_c \\ \xi - \xi_c \end{pmatrix} = A \begin{pmatrix} \delta\tau \\ \delta\xi \end{pmatrix}$$



LYZ Ratio for General CP

$\bar{y} = y_t - y_h = -0.894$

LYZ Ratio

$$R_{nm}(t) = \frac{\xi_{\mathrm{I}}^{(n)}(\tau)}{\xi_{\mathrm{I}}^{(m)}(\tau)} = \frac{X_n}{X_m}\left(1 + C\tau L^{y_t} + \mathcal{O}(\tau^2)\right)\left(1 + DL^{2\bar{y}} + \mathcal{O}(L^{4\bar{y}})\right)$$

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Binder cumulant

Jin+, PRD86, 2017

$$B_4(t) = b_4\left(1 + c\tau L^{y_t} + \mathcal{O}(t^2)\right)\left(1 + dL^{\bar{y}} + \mathcal{O}(L^{2\bar{y}})\right)$$

nonzero for $a_{12} \neq 0$

Deviation at $t = 0$ due to $a_{12} \neq 0$ converges faster in LYZ ratio.

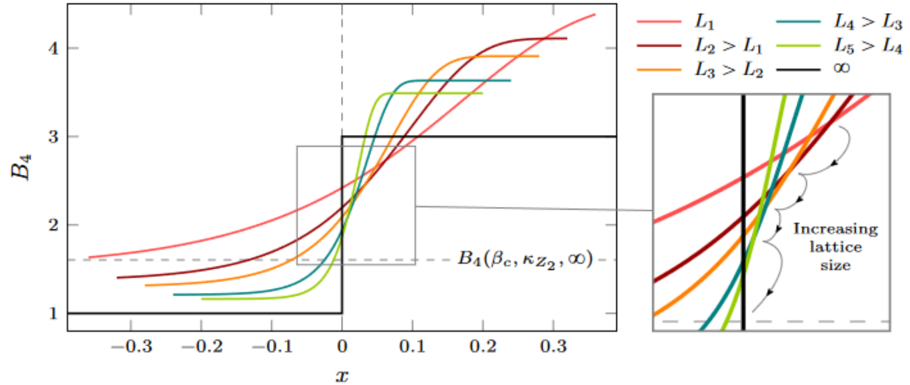


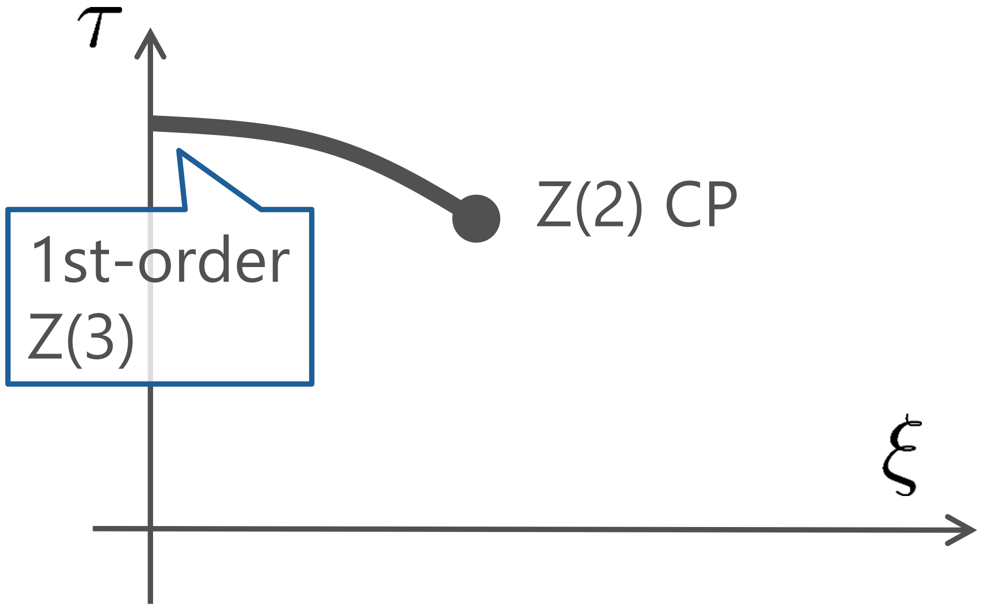
fig from Cuteri+, PRD ('21)

Numerical Analysis: 3d 3-State Potts Model

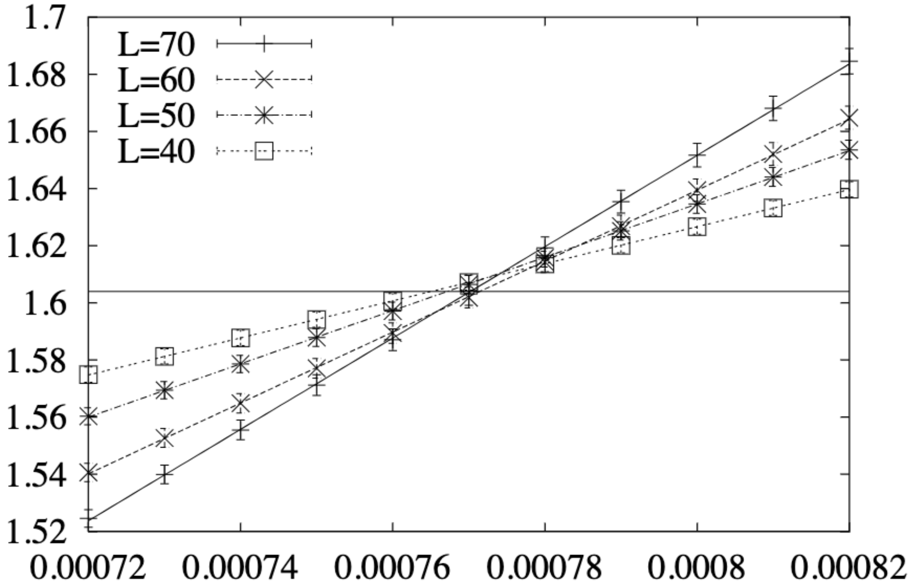
$$H = -\tau \sum_{\langle i,j \rangle} \delta_{\sigma_i,\sigma_j} - \xi \sum_i \delta_{\sigma_i,1} \quad \sigma_i = 1,2,3$$

Monte-Carlo + reweighting

Phase Diagram

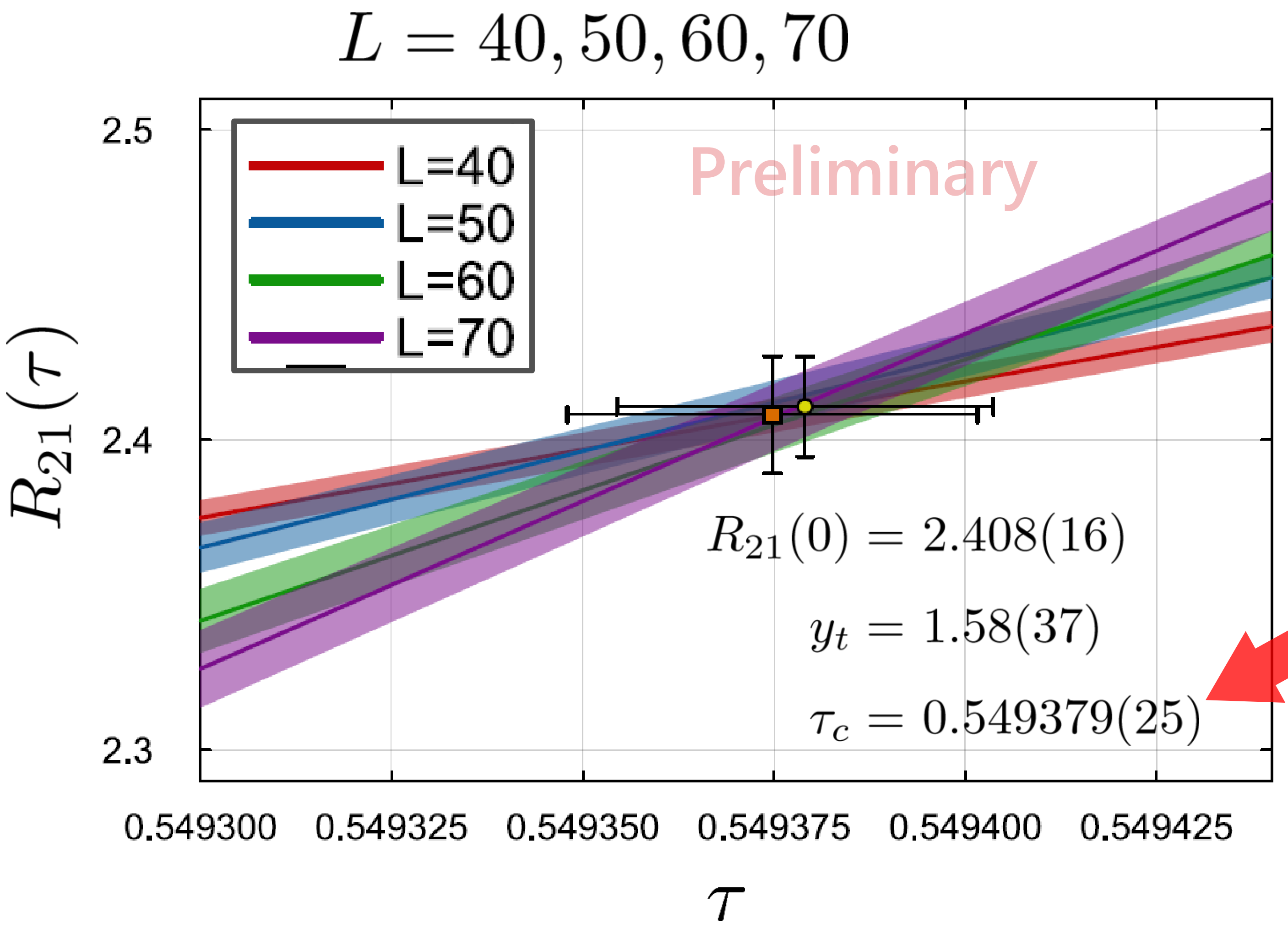


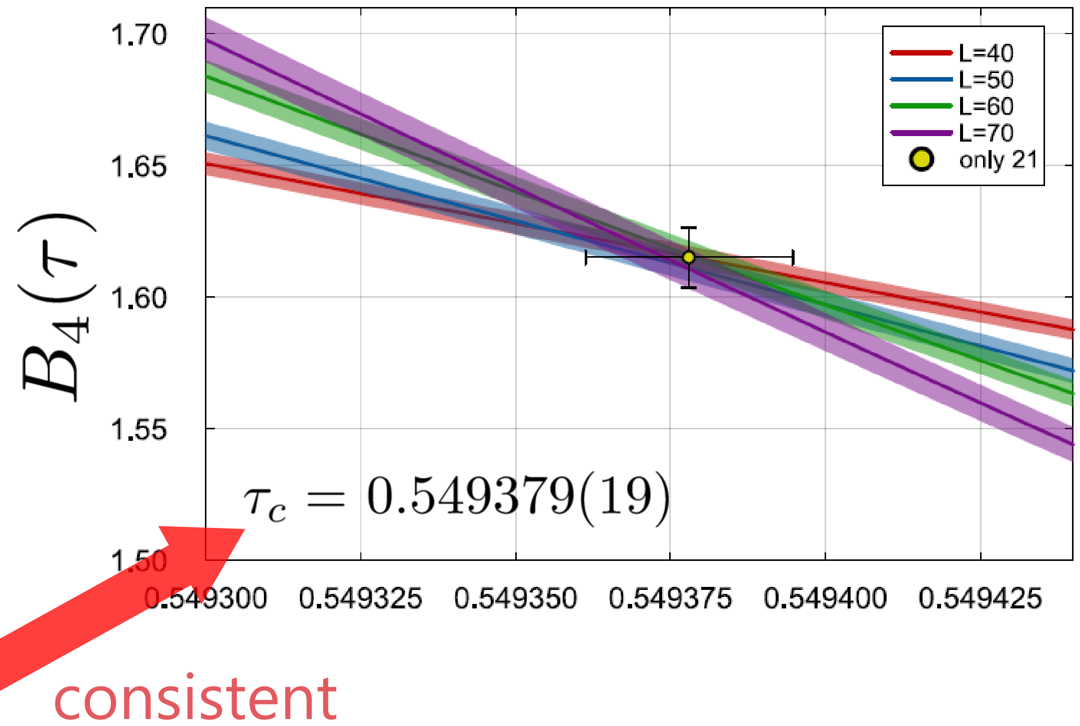
Binder-Cumulant Analysis



Karsch, Stickan, 2000

3d 3-State Potts Model: LYZ Ratio

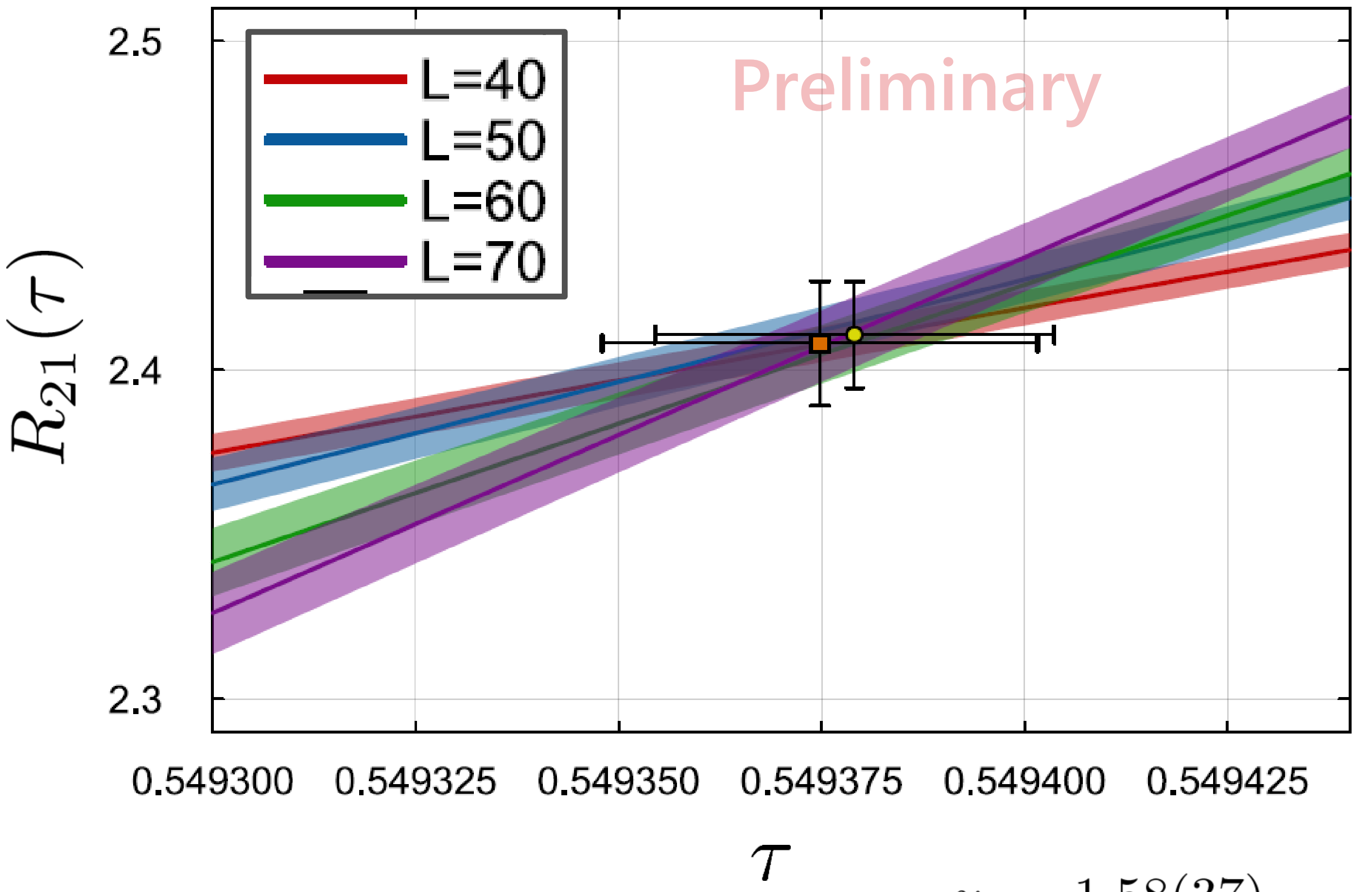


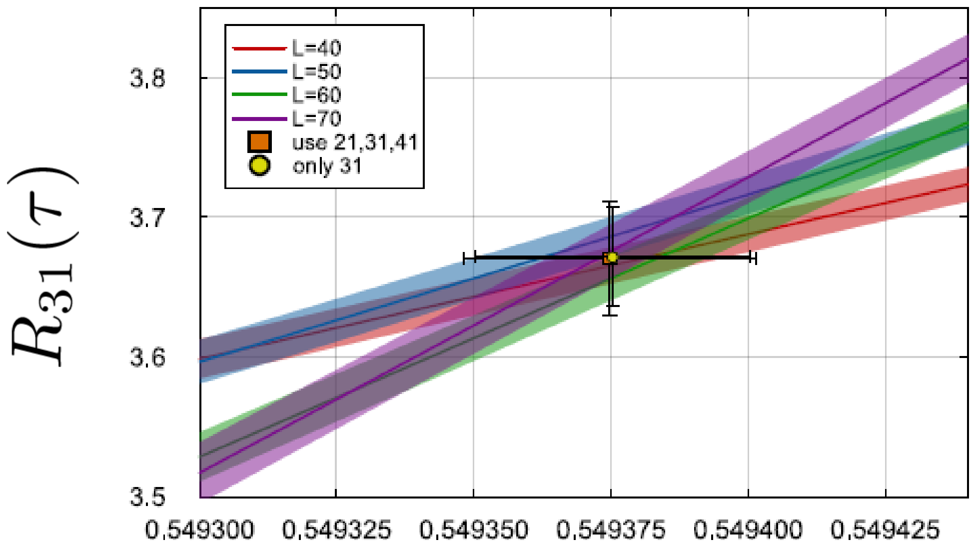


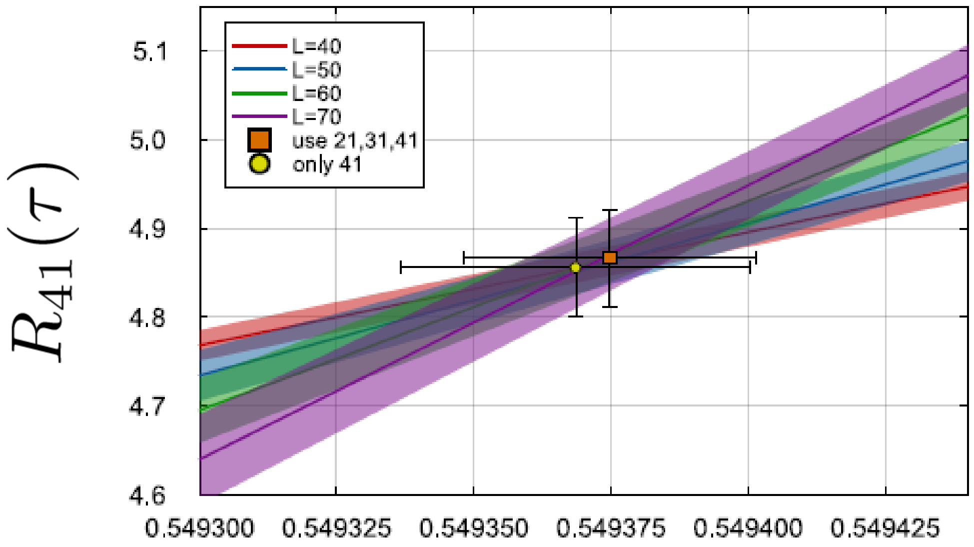
R_{n1} and B_4 give the same value of τ_c within statistics.

3d 3-State Potts Model: LYZ Ratio

$$L = 40, 50, 60, 70$$







$$y_t = 1.58(37)$$

$$\tau_c = 0.549375(24)$$

Summary

— LYZ give us valuable information of phase transitions.
— Their finite-size effects are non-negligible in typical numerical simulations.

$$R_{nm}(t) = \frac{h^{(n)}(t)}{h^{(m)}(t)}$$

— We proposed the use of the **ratio of LYZ** for locating a CP.
— Verification in 3d-Potts model.

Outlook

— Mapping of Ising variables
— Measurement of $R_{nm}(0)$ in Ising model

Question: Can lattice QCD find the 2nd LYZ??

backup

QCD-CP and LYZ

arXiv:2405.10196v1 [hep-lat] 16 May 2024

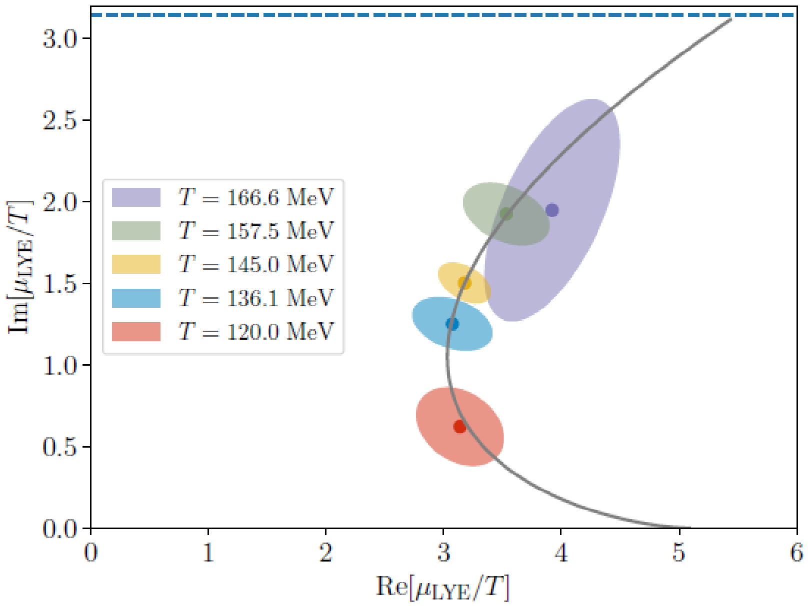


FIG. 3. Singularities at $T = 166.6, 157.5, 145.0, 136.1$ and 120.0 MeV. The dashed line lies at $\hat{\mu}_B = i\pi$.

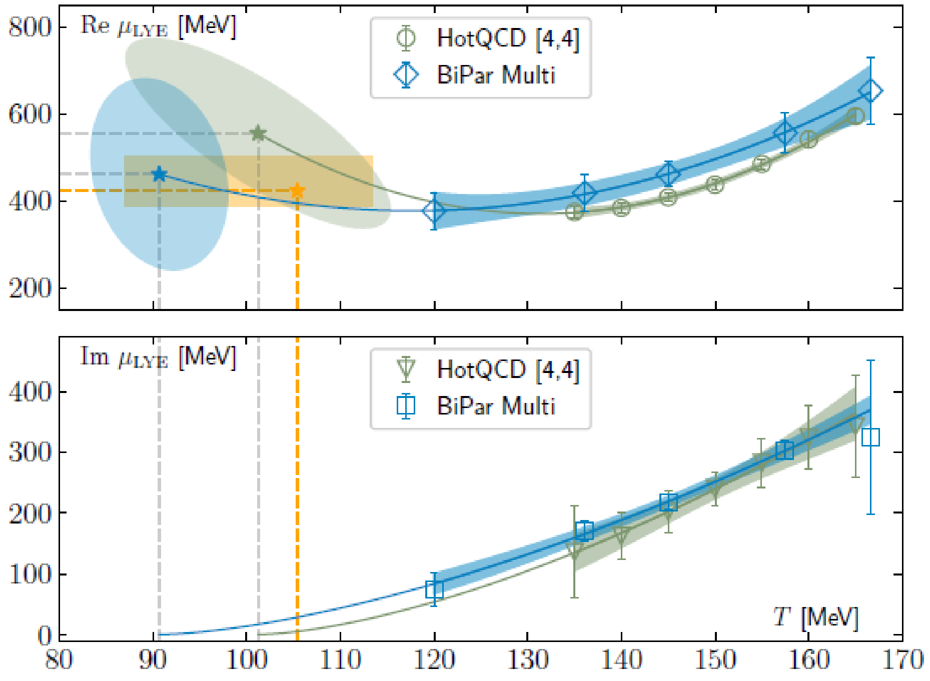
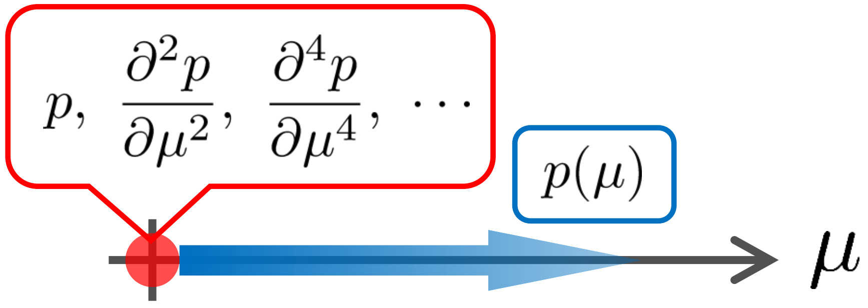


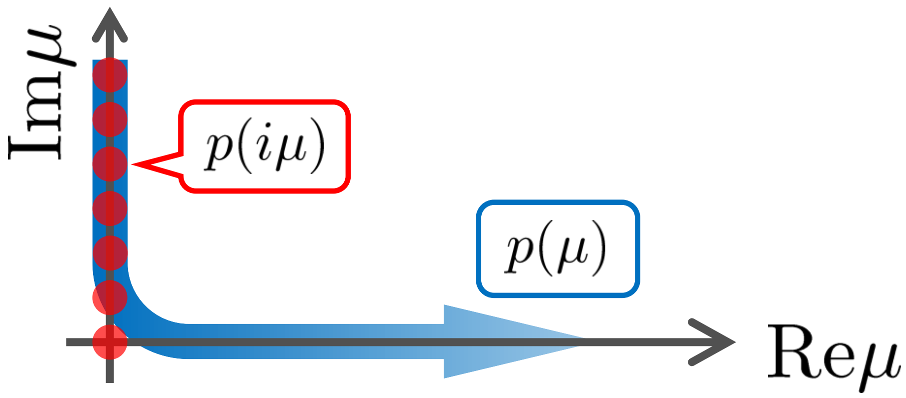
FIG. 4. Scaling fits for the LYE singularities related to the CEP. Green data come from a [4,4] Padé from Ref. [7]. Blue data come from the multi-point Padé. *Top*: Scaling of the real part. *Bottom*: Scaling of the imaginary part. The ellipses shown in the top panel represent the 68% confidence region deduced from the covariance matrix of the fit. The orange box indicates the AIC weighted estimate (6).

Using LYZ for the QCD-CP Search

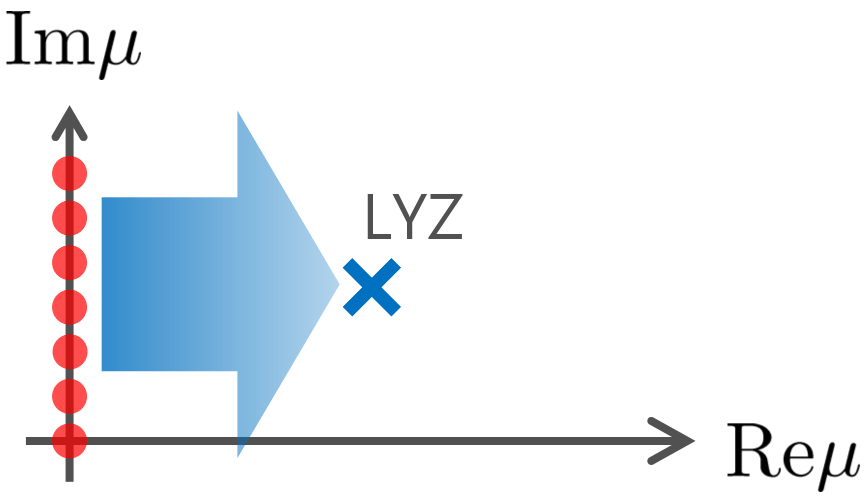
Taylor expansion



Imaginary chem. pot.



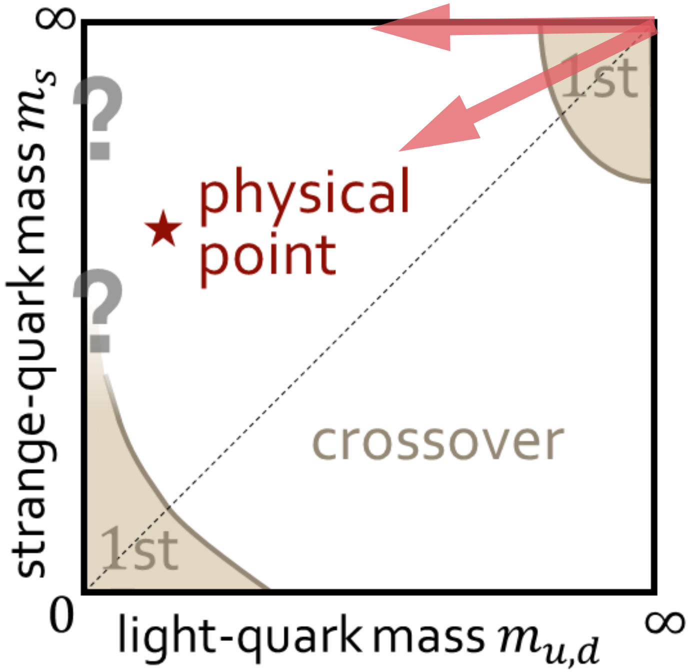
LYZ



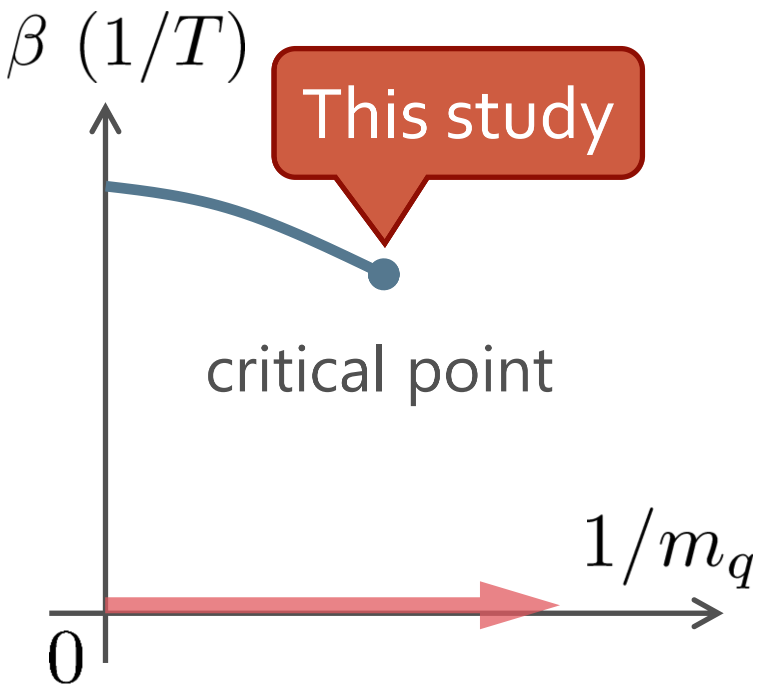
Use of Pade approximation

CP in Heavy-Quark QCD

Columbia Plot



Phase Diagram



CP in heavy-quark QCD

– $\mu_q = 0$ & large m_q

Easy to handle in lattice simulations!

We study the LYZ around the HQ-QCD-CP.

Hopping-Parameter Expansion (HPE)

$\sim 1/m_q$ expansion

Kiyohara, MK, Ejiri, Kanaya, PRD('21)
Ashikawa, MK, Ejiri, Kanaya, arXiv:2407.09156

Wilson Fermion

$$S_q = \sum_{x,y} \bar{\psi}_x M_{xy} \psi_y \qquad \kappa \sim \frac{1}{2m_q a}$$: hopping parameter

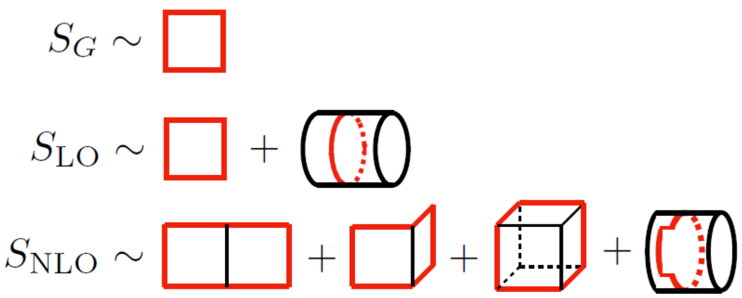
$$M_{xy} = \delta_{xy} - \kappa B_{xy} \qquad B_{xy} = \sum_{\mu=1}^{4} \left[(1-\gamma_\mu) U_{x,\mu} \delta_{y,x+\hat{\mu}} + (1+\gamma_\mu) U^\dagger_{y,\mu} \delta_{y,x-\hat{\mu}} \right]$$

nonzero only for neighboring (x, y)

Hopping-Parameer Expansion

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathcal{D}U_\mu \mathcal{O} e^{-S_g + \mathrm{tr} \ln M(\kappa)}$$

$$\mathrm{tr} \ln M(\kappa) = -\sum_{n=1}^{\infty} \frac{1}{n} \mathrm{tr}[B^n] \kappa^n$$



nth order terms in the HPE: closed trajectories of length n.

Higher-Order Terms in HPE

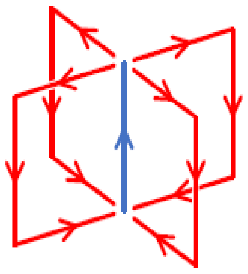
Kiyohara, MK, Ejiri, Kanaya, PRD('21)
Ashikawa, MK, Ejiri, Kanaya, arXiv:2407.09156

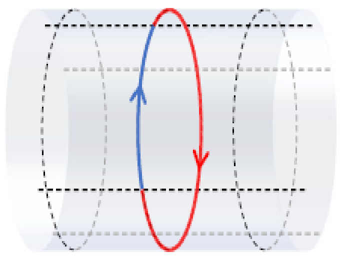
Monte Carlo Simulation @ LO

heat bath & over relaxation with modified staple

- Numerical cost is almost the same as the pure YM!

$$S_{\rm LO} = -6N_{\rm site}\beta^* \hat{P} - \lambda N_s^3 \hat{\Omega}_{\rm R}$$





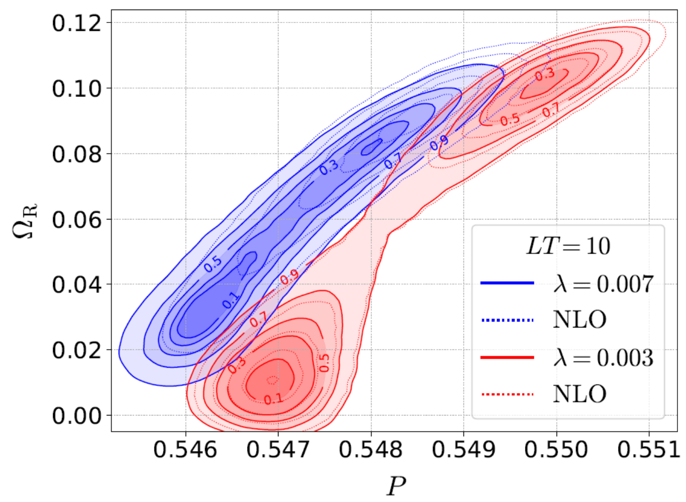
$\hat{P}$: plaquette
$\hat{\Omega}$: Polyakov loop
$\lambda = 2^{N_t+2} N_c \kappa^{N_t}$

NLO by Reweighting

$$\langle \mathcal{O} \rangle_{\rm NLO} = \frac{\langle \hat{O} e^{-S_{\rm NLO}} \rangle_{\rm LO}}{\langle e^{-S_{\rm NLO}} \rangle_{\rm LO}}$$

Overlapping problem is well suppressed due to the LO confs.

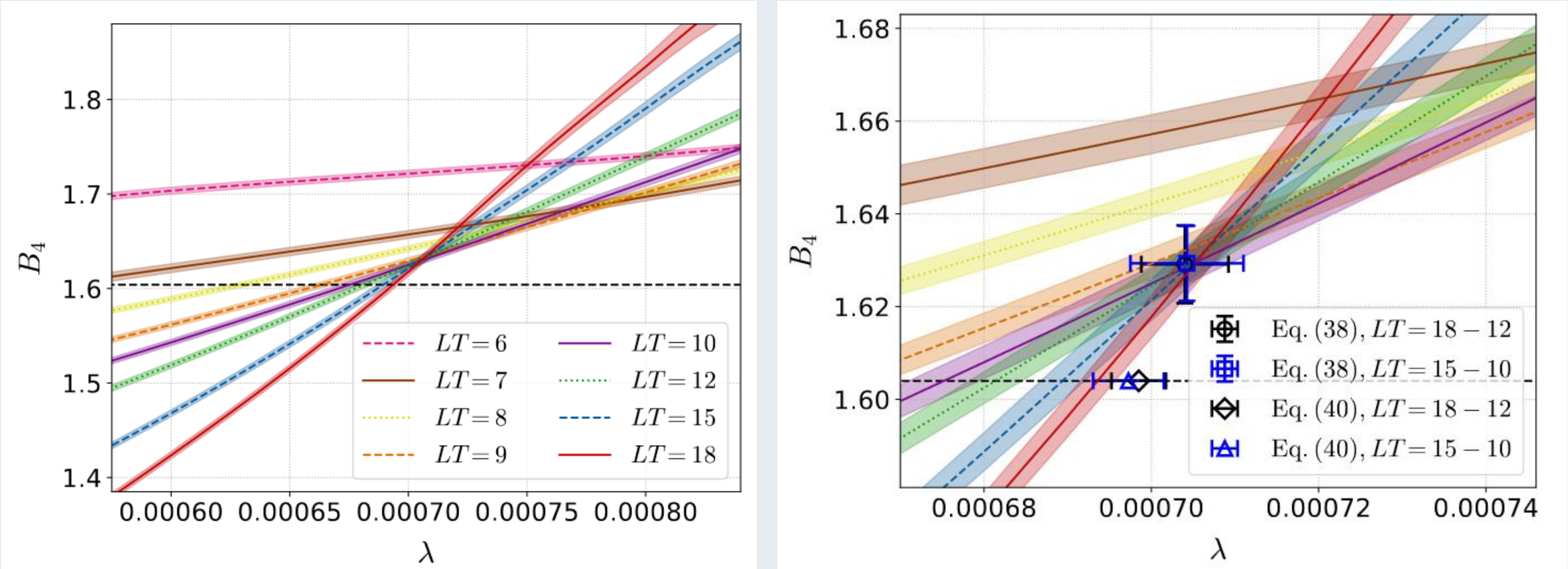
- Realize high statistical analysis



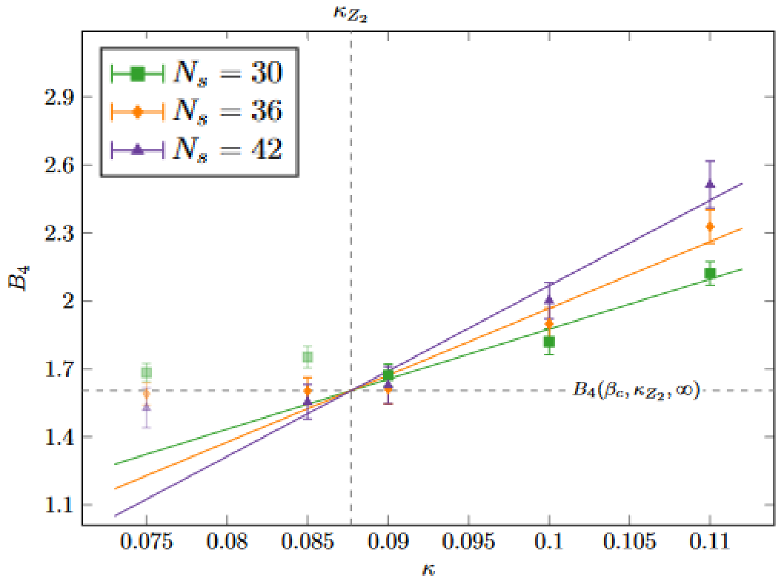
Binder Cumulant Analysis

$$LT = N_x/N_t$$
$$\lambda = 2^{N_t+2} N_c \kappa^{N_t}$$

$N_t = 4$: Kiyohara, MK, Ejiri, Kanaya, PRD, 2021
$N_t = 6$: Ashikawa, MK, Ejiri, Kanaya, arXiv:2407.09156



w/ Dynamical Fermions



Cuteri , Philipsen , Schön, Sciarra, ’21

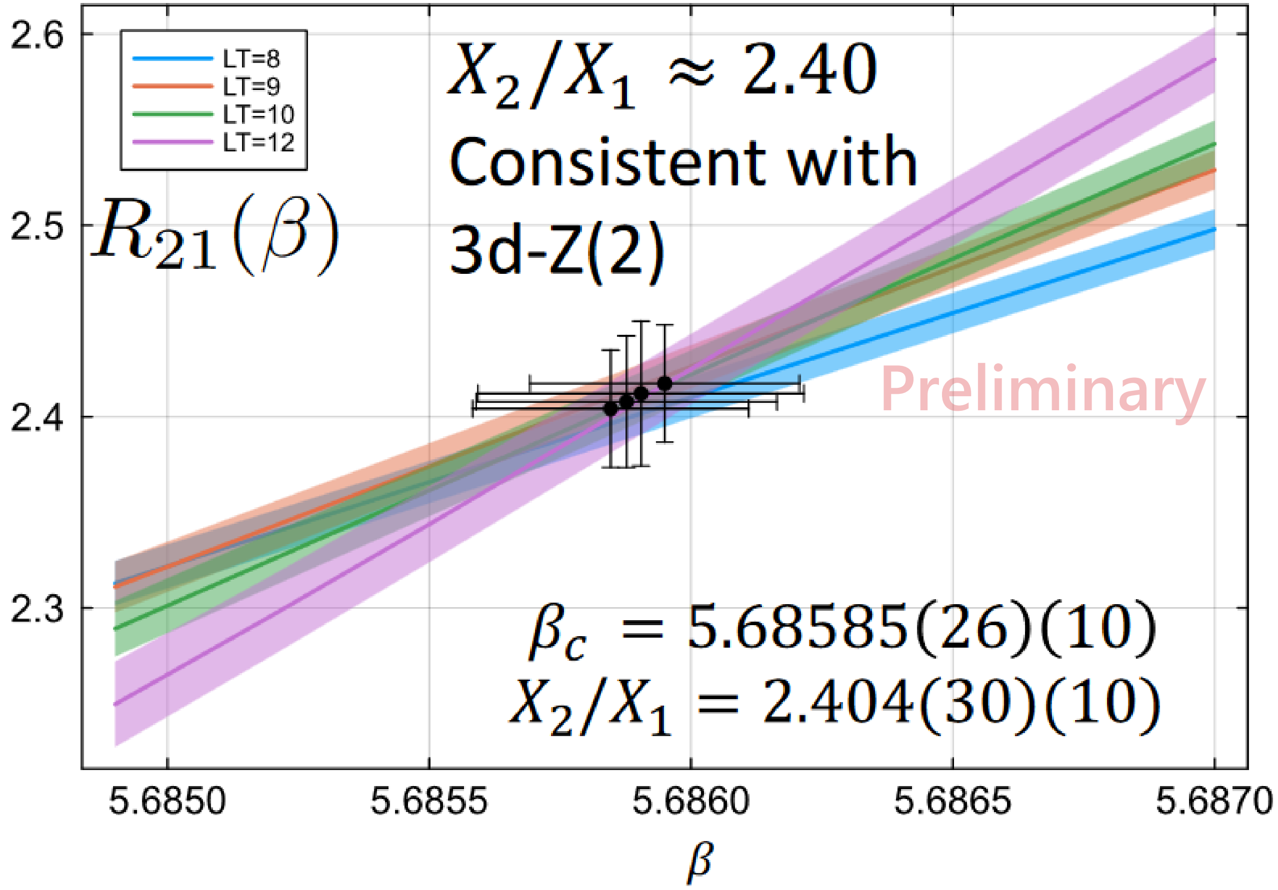
One order smaller statistical errors on more than twice larger LT!

Precise determination of the location of the CP

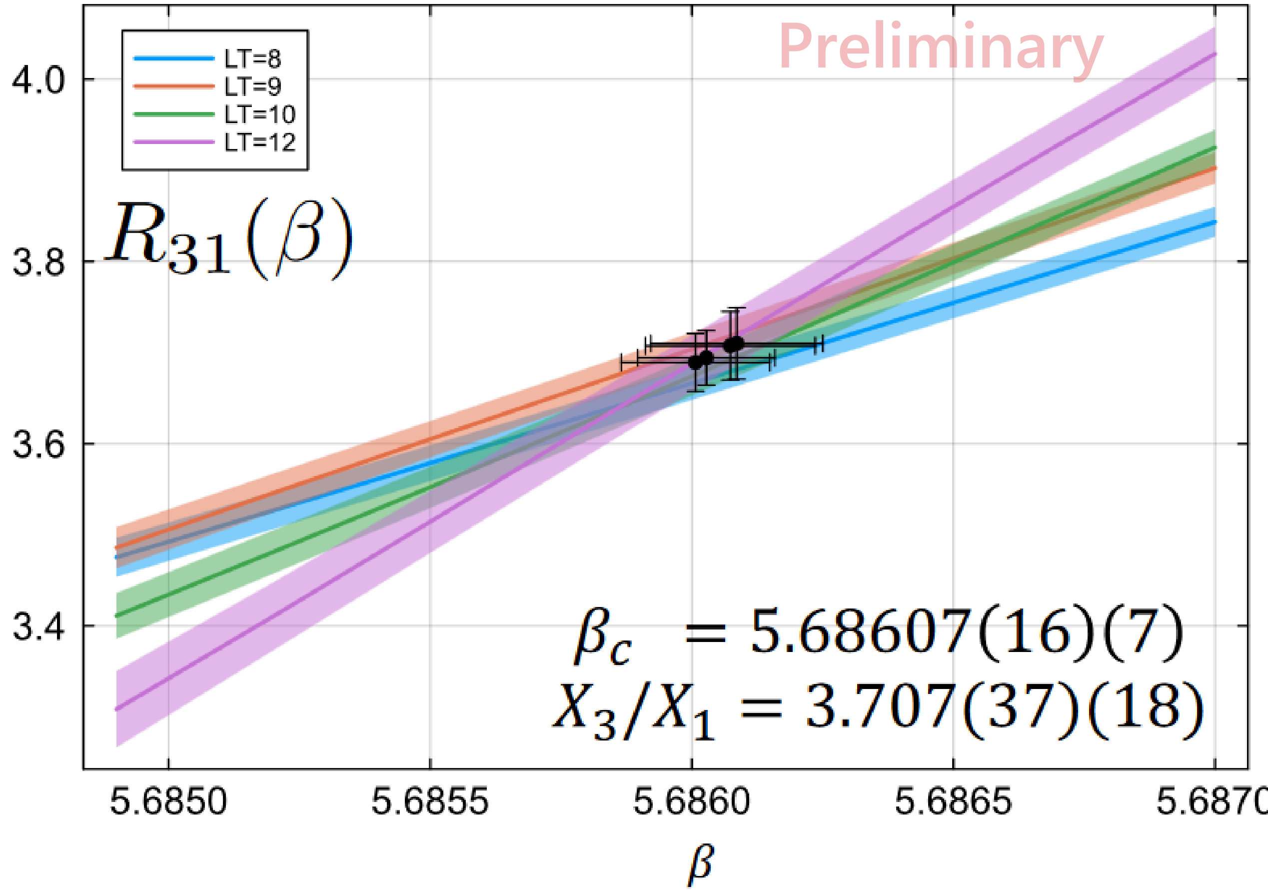
LYZ Ratio

$N_t = 4$, HPE-NLO

$2^{nd}/1^{st}$ LYZ Ratio



$3^{rd}/1^{st}$ LYZ Ratio



Conistent with the Binder-cumulant analysis $\boldsymbol{\beta_c = 5.68578(22)}$.