Phase structure of finite temperature-density QCD in heavy quark regime by hopping parameter expansion

Shinji Ejiri (Niigata University)

Collaborators:

Kazuyuki Kanaya (Univ. Tsukuba)

Masakiyo Kitazawa (YITP, Kyoto Univ.)

Masanari Koiida (Niigata Univ.)

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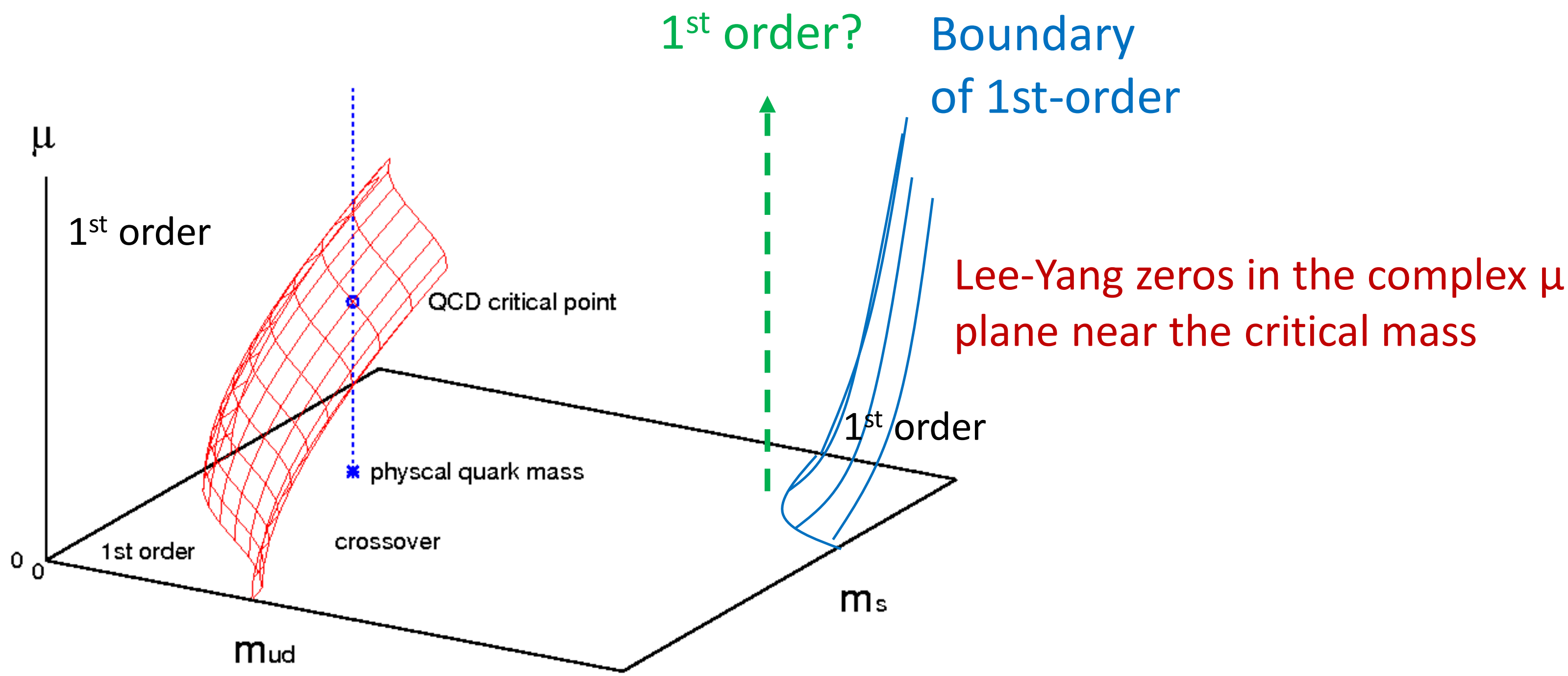
QCD phase diagram in the heavy quark region



- Applications of the effective heavy quark theory based on the hopping parameter expansion.
- Boundary of first-order transition in the heavy quark region.
  - First-order region: narrower as μ
- Singularities in the complex μ plane: Lee-Yang zeros $Z(\mu) = 0$
- Appearance of first-order phase transitions in the heavy and dense region.

Effective theory based on the hopping parameter expansion

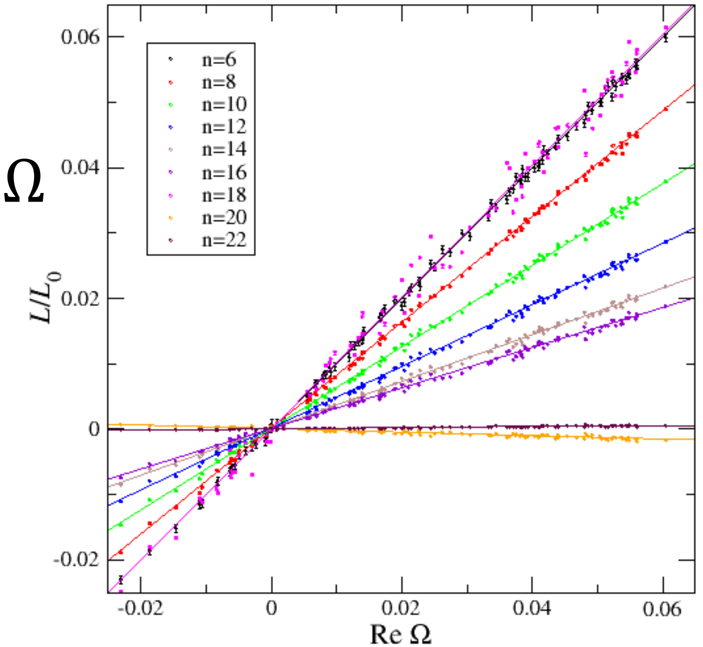
- We expand the quark determinant in terms of the hopping parameter κ.
- The terms that wind around the periodic boundary in the time direction are important.

$$L(N_t,n) = \sum_m L_m(N_t,n) = \sum_m\left(L_m^+(N_t,n) + L_m^-(N_t,n)\right)$$

- Higher order expansion terms $L(N_t,n)$ are very strongly correlated with the leading term: Polyakov loop Ω.

$$L(N_t,n) \approx L^0(N_t,n)c_n \,\mathrm{Re}\Omega, \quad \mathrm{Arg}\, L_1^+(N_t,n) \approx \mathrm{Arg}\,\Omega$$

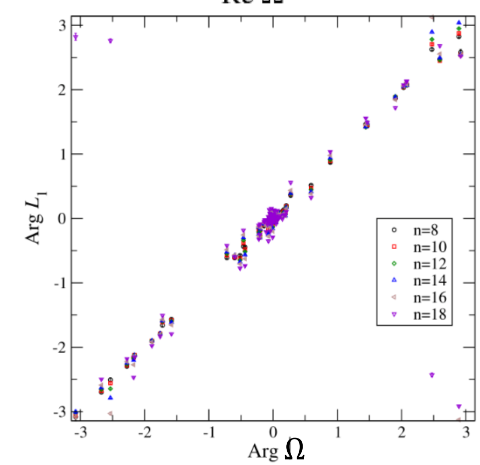
(PTEP 2022, 033B05) (arXiv:2311.11508)



- $L_1(N_t,n)$ is dominant: $L_1(N_t,n) \approx L(N_t,n)$.
- Effective action in the heavy quark region

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - \frac{N_s^3\lambda}{2}\left(\mathrm{e}^{\frac{\mu}{T}}\Omega + \mathrm{e}^{-\frac{\mu}{T}}\Omega^*\right)$$

$$\lambda = N_t \sum_{\mathrm{f}=1}^{N_\mathrm{f}} \sum_{n=N_t}^{n_{\max}} L^0(N_t,n)\, c_n \kappa_\mathrm{f}^n$$



- Even if the number of expansion terms increases significantly, the effects of higher-order terms can be incorporated.
- Since the calculation cost can be dramatically reduced, calculations with high accuracy are possible.

Boundary of first-order transition in the heavy quark region

Effective action in the heavy quark region

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - \frac{N_s^3 \lambda}{2}\left(\mathrm{e}^{\frac{\mu}{T}}\Omega + \mathrm{e}^{-\frac{\mu}{T}}\Omega^*\right), \qquad \lambda = N_t \sum_{\mathrm{f}=1}^{N_{\mathrm{f}}} \sum_{n=N_t}^{n_{\mathrm{max}}} L^0(N_t, n)\, c_n \kappa_{\mathrm{f}}^n$$

Ignoring the effect of complex phase,

$$S_{\mathrm{eff}} \approx -6N_{\mathrm{site}}\beta^* P - N_s^3 \lambda \cosh\frac{\mu}{T} \mathrm{Re}\Omega$$

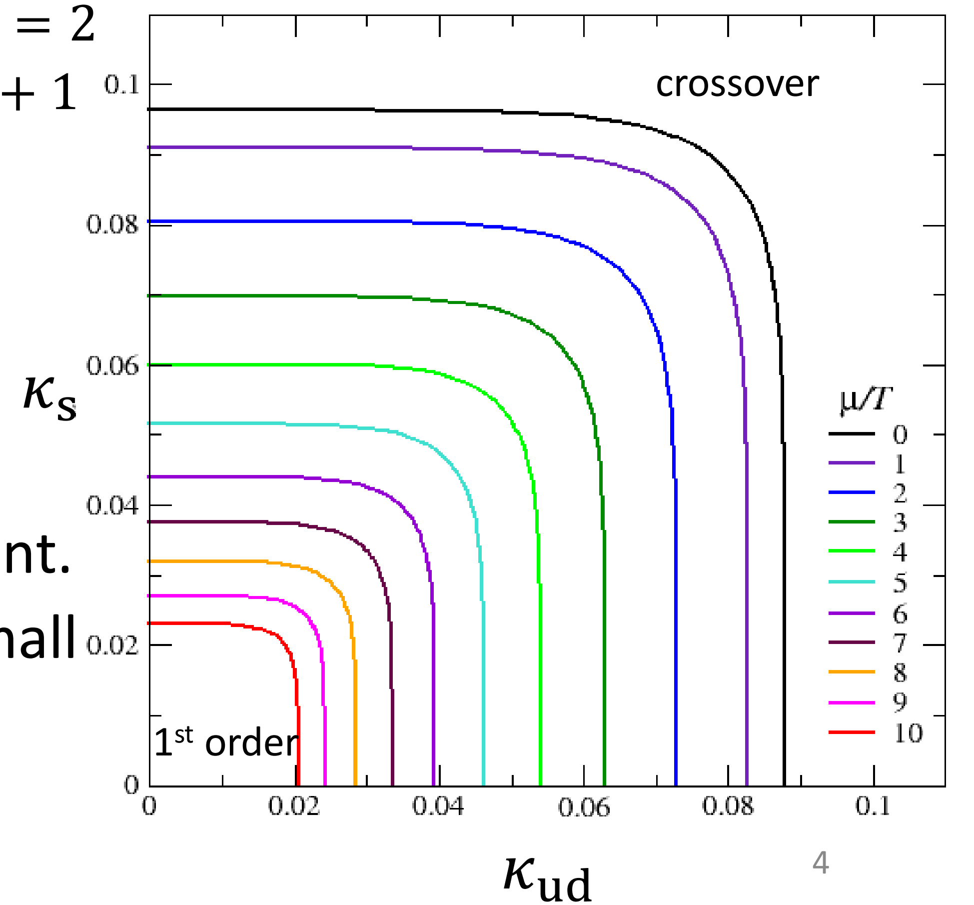
- Once the critical point λ_c at $\mu = 0$ for $N_{\mathrm{f}} = 2$ is determined, the critical line for $N_{\mathrm{f}} = 2+1$ can be given by solving

$$\lambda_c = N_t \cosh\frac{\mu}{T} \sum_{\mathrm{f}=1}^{N_{\mathrm{f}}} \sum_{n=N_t}^{n_{\mathrm{max}}} L^0(N_t, n)\, c_n \kappa_{\mathrm{f}}^n$$

Adding the effect of complex phase,

- Sign problem is mild near the critical point.
    - Effect of the complex phase: very small
- As increasing density, the first-order transition region becomes narrower.

$N_t = 6$ (arXiv:2311.11508)



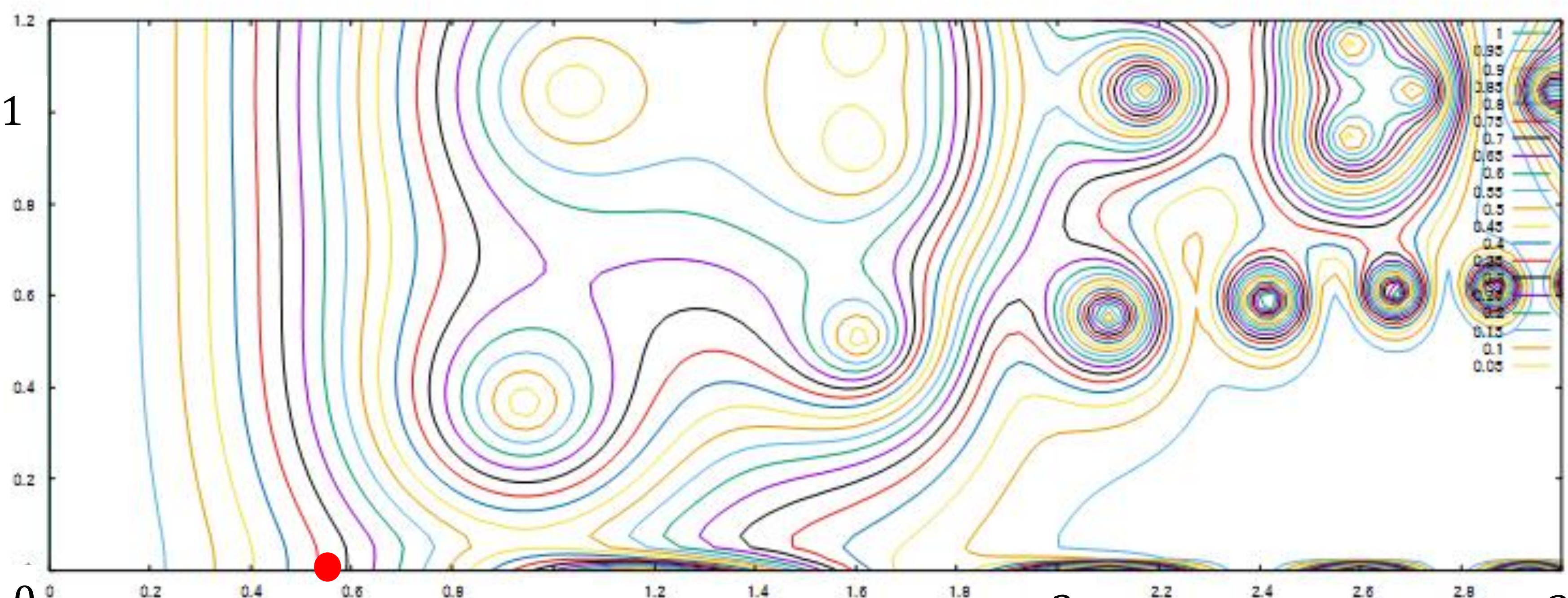
Lee-Yang zeros in the complex μ plane near the critical mass

$$Z(\beta,\lambda,\mu) = \int DU\ e^{-S_{\mathrm{eff}}} \qquad (\mu = \mu_R + i\mu_I)$$

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - \frac{N_S^3\lambda}{2}\left(\mathrm{e}^{\frac{\mu}{T}}\Omega + \mathrm{e}^{-\frac{\mu}{T}}\Omega^*\right)$$

$$= -6N_{\mathrm{site}}\beta^* P - N_S^3\lambda\left[\cosh\frac{\mu_R}{T}\left(\cos\frac{\mu_I}{T}\ \Omega_R - \sin\frac{\mu_I}{T}\ \Omega_I\right) + i\sinh\frac{\mu_R}{T}\left(\sin\frac{\mu_I}{T}\ \Omega_R + \cos\frac{\mu_I}{T}\ \Omega_I\right)\right]$$

- Indicator of Lee-Yang zero: $\frac{|Z(\beta,\lambda,\mu)|}{Z_{\mathrm{pq}}(\beta,\lambda,\mu)}$ (Z_{pq}: Ignoring the imaginary part of S_{eff})
- Reweighting method $N_t = 6, N_s = 90, \beta^* = 5.8905, \lambda = 0.0010$ (simulation pomt: $\lambda_0 = 0.0012$)



Data: Ashikawa et al., arXiv2407.09156

Lee-Yang zeros in the complex μ plane near the critical mass

$$Z(\beta,\lambda,\mu) = \int DU\ e^{-S_{\rm eff}} \qquad (\mu = \mu_R + i\mu_I)$$

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - \frac{N_s^3\lambda}{2}\left(e^{\frac{\mu}{T}}\Omega + e^{-\frac{\mu}{T}}\Omega^*\right)$$

$$= -6N_{\rm site}\beta^* P - N_s^3\lambda\left[\cosh\frac{\mu_R}{T}\left(\cos\frac{\mu_I}{T}\,\Omega_R - \sin\frac{\mu_I}{T}\,\Omega_I\right) + i\sinh\frac{\mu_R}{T}\left(\sin\frac{\mu_I}{T}\,\Omega_R + \cos\frac{\mu_I}{T}\,\Omega_I\right)\right]$$

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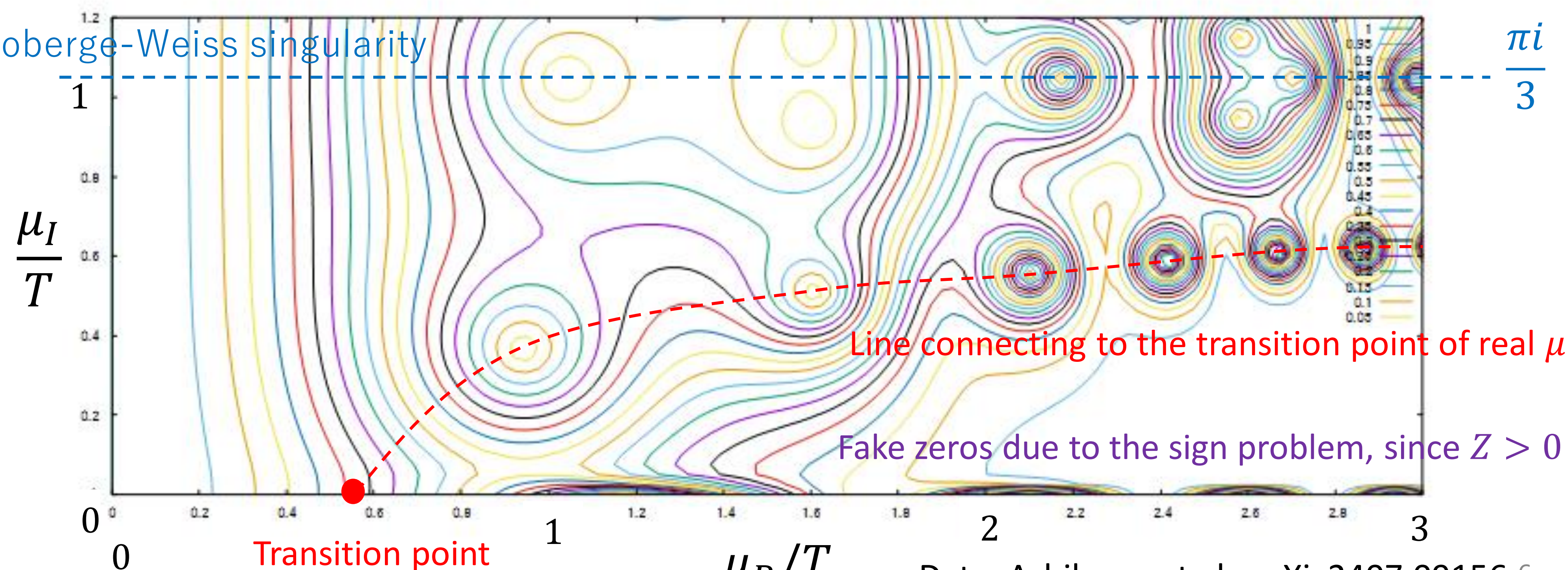

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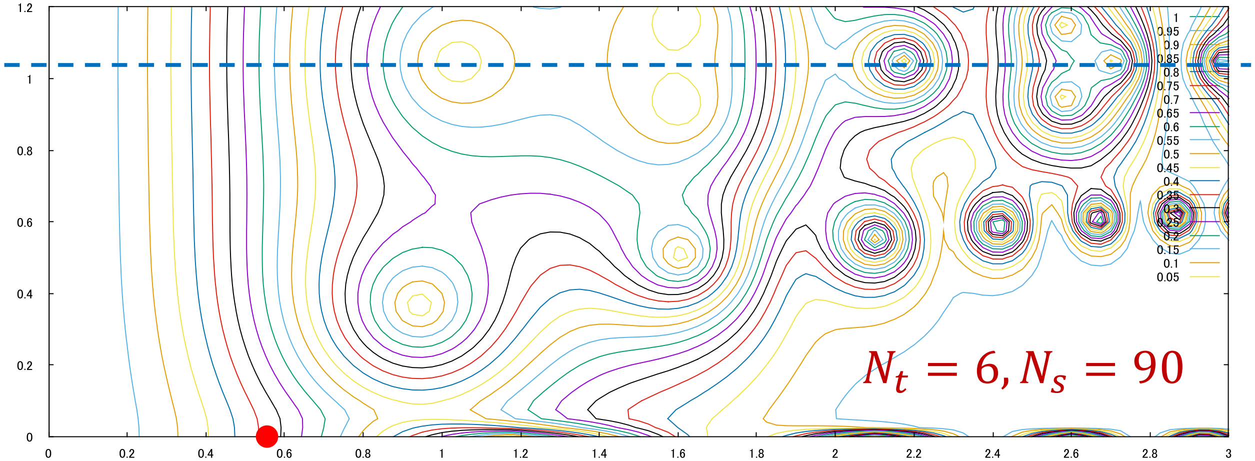
Lee-Yang zeros in Phase quenched (isospin) QCD

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - \frac{N_S^3\lambda^*}{2}\left(e^{\frac{\mu}{T}}\Omega + e^{-\frac{\mu}{T}}\Omega^*\right) \to \underline{-6N_{\rm site}\beta^* P - N_S^3\lambda \cosh\frac{\mu}{T}\Omega_{\rm R}}$$

Complex phase: ignored



Similar to 3-state Potts model

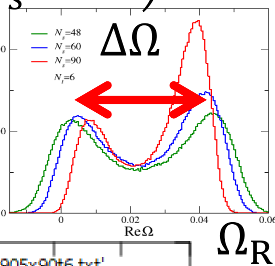
Critical point: ising universality class

Lee-Yang zero for

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - N_S^3\lambda\Omega_{\rm R}$$

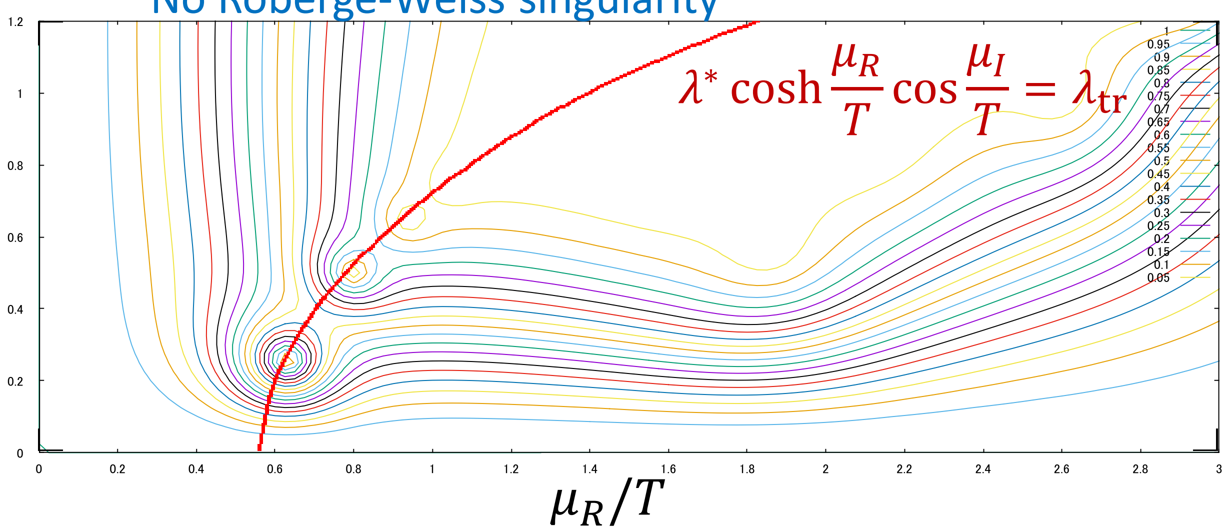
$$(\lambda_{\rm R}^{\rm LYZ}, \lambda_{\rm I}^{\rm LYZ}) = \left(\lambda_{\rm tr}, \frac{(2n+1)\pi}{N_S^3\Delta\Omega}\right)$$

($\lambda_{\rm tr}$: transition point for real λ, n: integer.)

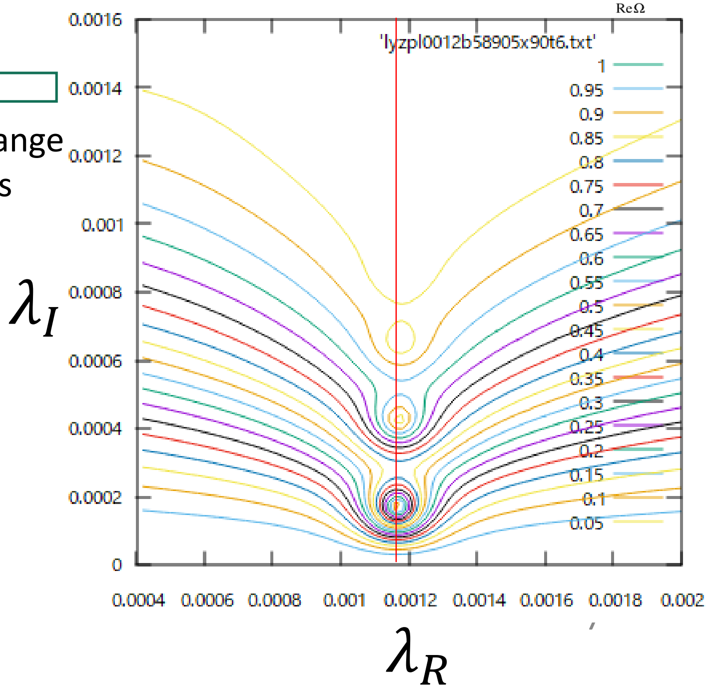


$$S_{\rm eff} = \cdots - N_S^3\lambda^*\left[\cosh\frac{\mu_R}{T}\left(\cos\frac{\mu_I}{T}\Omega_R - \sin\frac{\mu_I}{T}\Omega_I\right) + i\sinh\frac{\mu_R}{T}\left(\sin\frac{\mu_I}{T}\Omega_R + \cos\frac{\mu_I}{T}\Omega_I\right)\right]$$

No Roberge-Weiss singularity



change axis



- The distance from the real axis is larger than expected.
- The transition point on the real axis is smaller than $\mu_R^{\rm LYZ}$.

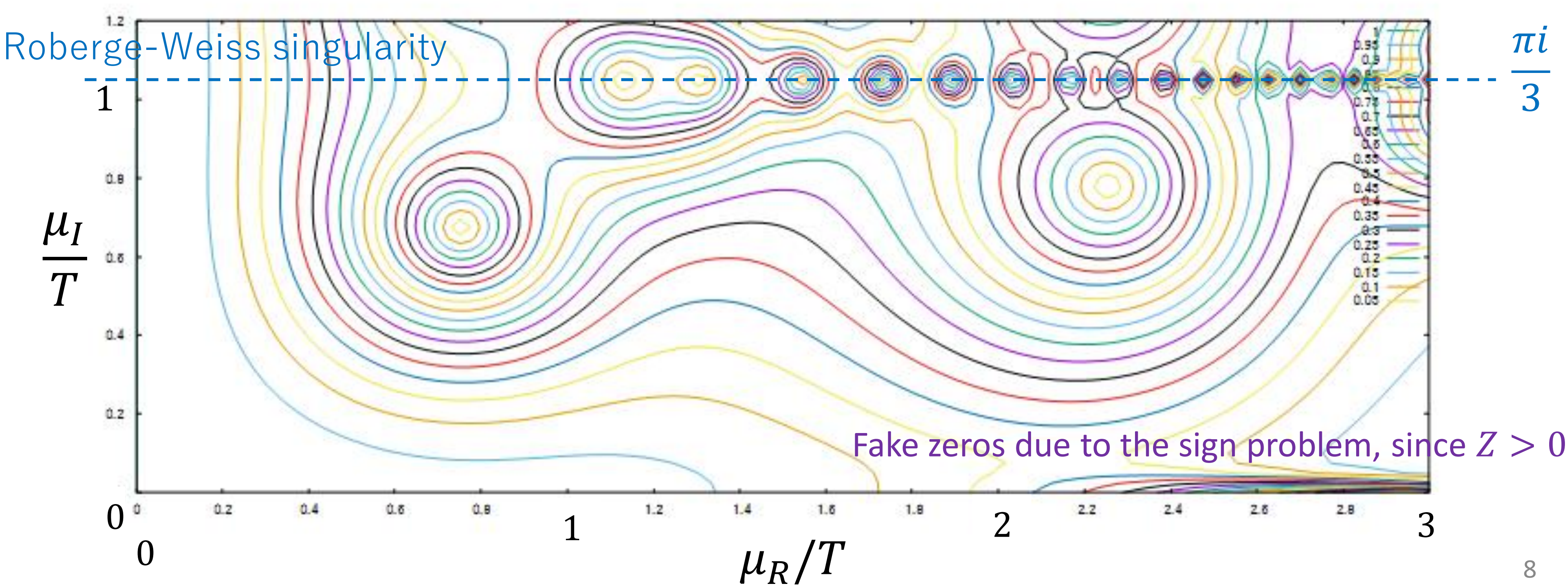
Lee-Yang zeros in the complex μ plane when Ns is small

$$Z(\beta,\lambda,\mu) = \int DU\ e^{-S_{\rm eff}} \qquad (\mu = \mu_R + i\mu_I)$$

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - \frac{N_s^3\lambda}{2}\left(e^{\frac{\mu}{T}}\Omega + e^{-\frac{\mu}{T}}\Omega^*\right)$$

$$= -6N_{\rm site}\beta^* P - N_s^3\lambda\left[\cosh\frac{\mu_R}{T}\left(\cos\frac{\mu_I}{T}\ \Omega_R - \sin\frac{\mu_I}{T}\ \Omega_I\right) + i\sinh\frac{\mu_R}{T}\left(\sin\frac{\mu_I}{T}\ \Omega_R + \cos\frac{\mu_I}{T}\ \Omega_I\right)\right]$$

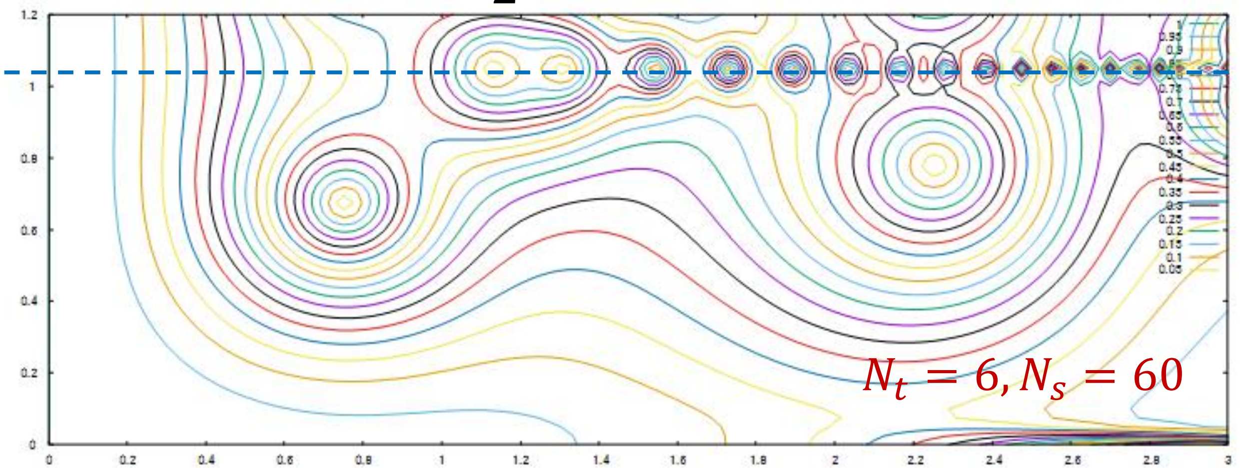
- Indicator of Lee-Yang zero: $\dfrac{|Z(\beta,\lambda,\mu)|}{Z_{\rm pq}(\beta,\lambda,\mu)}$ ($Z_{\rm pq}$: Ignoring the imaginary part of $S_{\rm eff}$)
- Reweighting method $N_t = 6, N_s = 60, \beta^* = 5.8911, \lambda = 0.0010$ (simulation pomt: $\lambda_0 = 0.0010$)



Lee-Yang zeros in Phase quenched (isospin) QCD

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - \frac{N_s^3\lambda}{2}\left(e^{\frac{\mu}{T}}\Omega + e^{-\frac{\mu}{T}}\Omega^*\right) \rightarrow \underline{-6N_{\rm site}\beta^* P - N_s^3\lambda\cosh\frac{\mu}{T}\Omega_{\rm R}}$$

Complex phase: ignored



Similar to 3-state Potts model

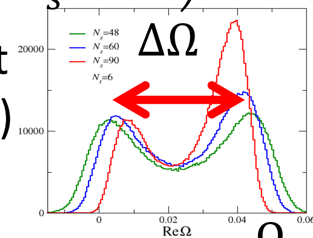
Critical point: ising universality class

Lee-Yang zero for

$$S_{\rm eff} = -6N_{\rm site}\beta^* P - N_s^3\lambda\Omega_{\rm R}$$

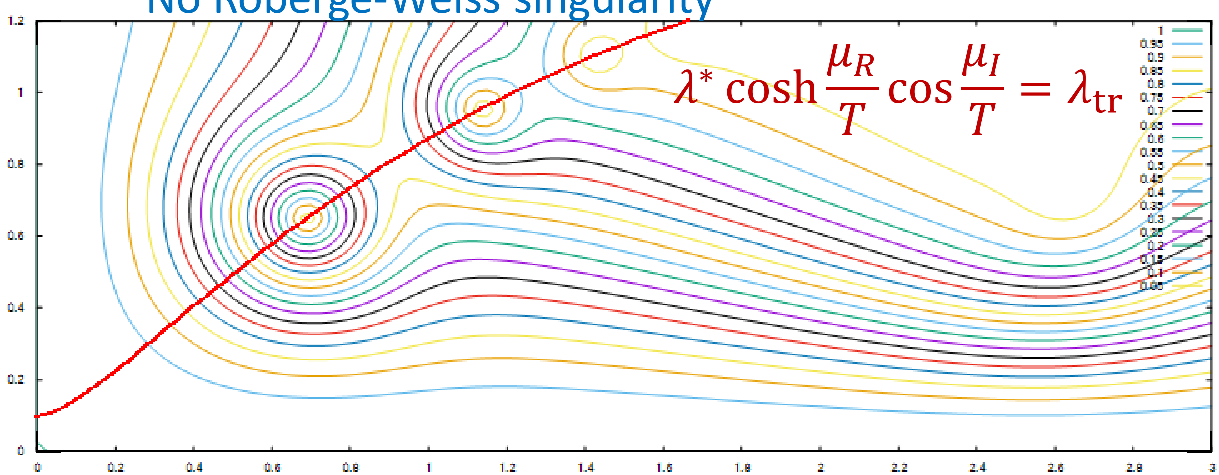
$$\left(\lambda_{\rm R}^{\rm LYZ}, \lambda_{\rm I}^{\rm LYZ}\right) = \left(\lambda_{\rm tr}\,, \frac{(2n+1)\pi}{N_s^3\Delta\Omega}\right)$$

($\lambda_{\rm tr}$: transition point for real λ, n: integer.)



$$S_{\rm eff} = \cdots - N_s^3\lambda^*\left[\cosh\frac{\mu_R}{T}\left(\cos\frac{\mu_I}{T}\Omega_R - \sin\frac{\mu_I}{T}\Omega_I\right) + i\sinh\frac{\mu_R}{T}\left(\sin\frac{\mu_I}{T}\Omega_R + \cos\frac{\mu_I}{T}\Omega_I\right)\right]$$

No Roberge-Weiss singularity



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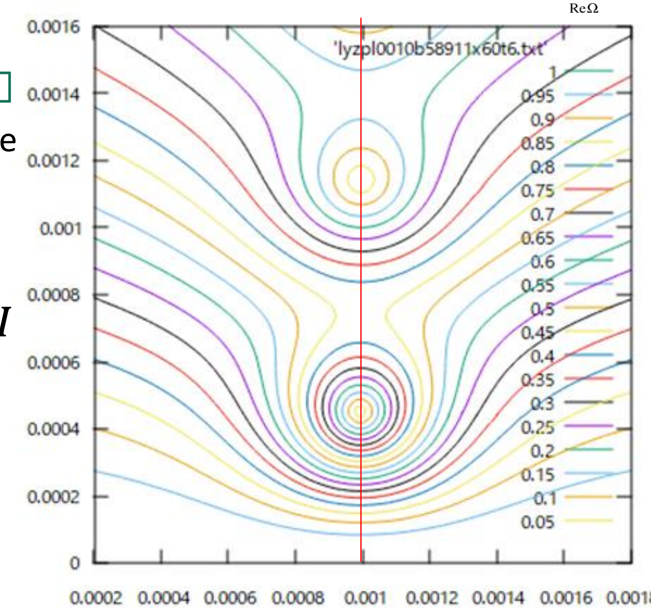
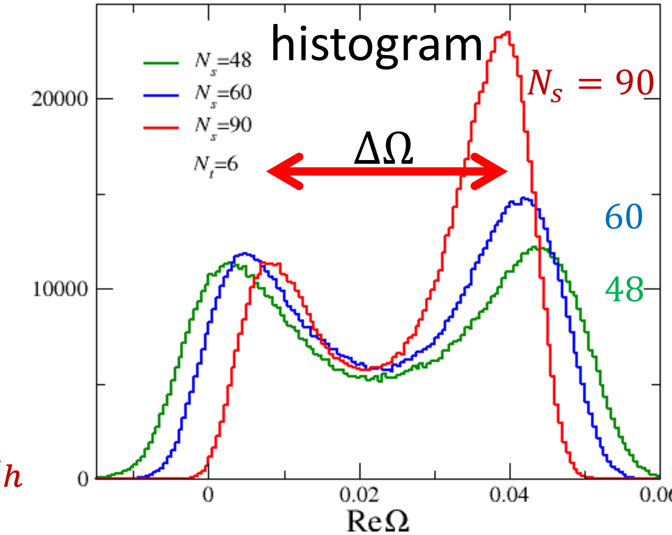


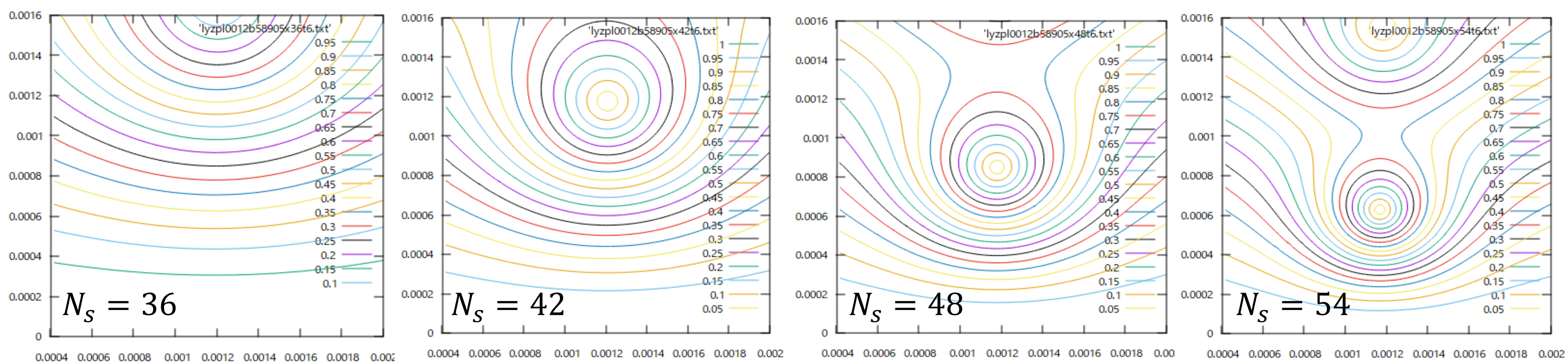
- If the volume is small, the L-Y zeros become indistinguishable from the R-W singularity.

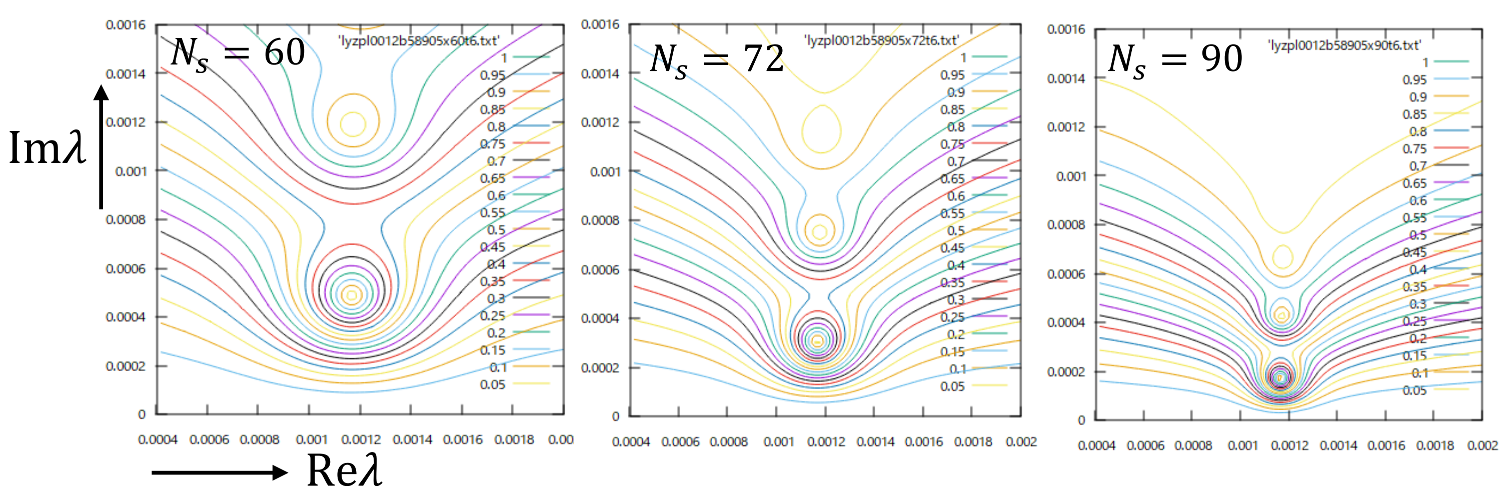
Lee-Yang zero in heavy quark QCD

- Action: $S_{\text{eff}} = -6N_{\text{site}}\beta P - N_s^3 \lambda \Omega_{\text{R}}$ $\quad (\lambda = \lambda_R + i\lambda_I)$
- Indicator of L-Y zero: $Z_{\text{nor}}(\beta,\lambda) = \dfrac{|Z(\beta,\lambda)|}{Z(\beta,\lambda_R)}$ $\qquad$ (n: integer.)
- Conditions at LYZ: $\underline{\lambda_R = (\text{const}),\ \ N_s^3 \Delta\Omega \lambda_I = (2n+1)\pi}$
- At the critical point, $\Delta\Omega \sim N_s^{-3+y_h}$, $\qquad \lambda_I = \dfrac{(2n+1)\pi}{N_s^3 \Delta\Omega} \sim N_s^{-y_h}$

  ising universality class : $y_h = 2.482$







Simulation near the critical point

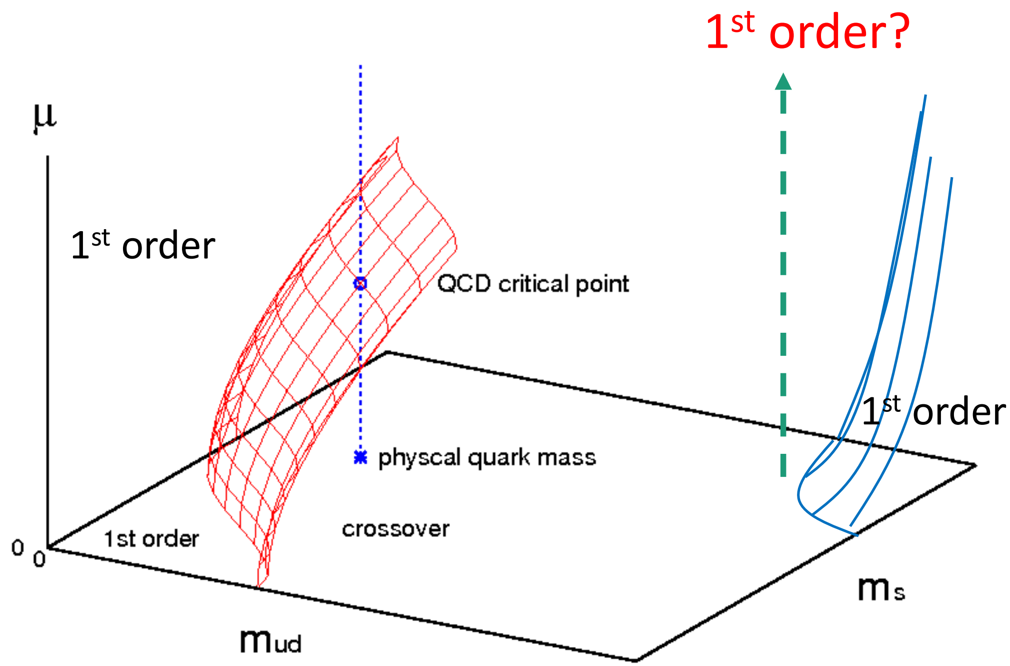
$\beta = 5.8905$

$\lambda = 0.0012$

$N_t = 6$

$N_s = 36 - 90$

Boundaries of first-order transitions in QCD phase diagram



- We expect that the first-order phase transition in the light mass region expands with increasing density.
- The first-order phase transition region may expand into the heavy quark region.

In the heavy-dense effective theory $(\kappa \rightarrow 0, \mathrm{e}^{\mu/T} \rightarrow \infty, \mathrm{e}^{-\mu/T} \rightarrow 0)$

- Parameter: $C = \kappa^{N_t} \mathrm{e}^{\mu/T}$. Symmetry under $C \rightarrow 1/C$ and $\Omega \rightarrow \Omega^*$
- In the heavy limit, [small μ] 1st order → crossover → 1st order [large μ]

Finite density Effect

$$S_{\mathrm{eff}} = -6N_{\mathrm{site}}\beta^* P - N_S^3 \lambda \left(\cosh\frac{\mu}{T}\,\Omega_{\mathrm{R}} + i\sinh\frac{\mu}{T}\,\Omega_{\mathrm{I}} \right) \qquad (\Omega = \Omega_{\mathrm{R}} + i\Omega_{\mathrm{I}})$$

- Partition function:

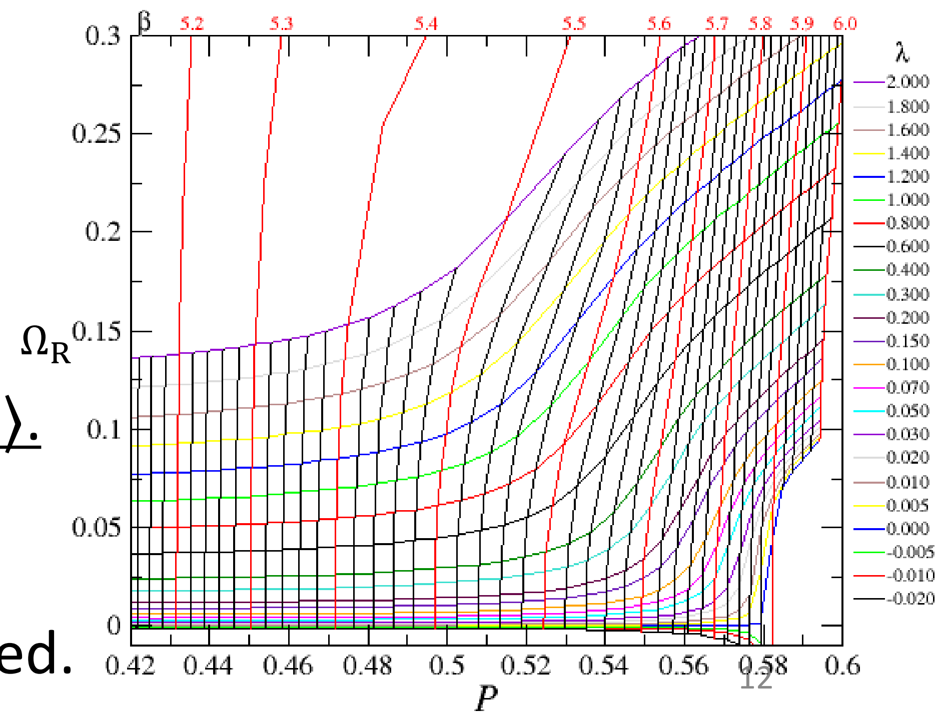
$$Z = \int DU\ e^{-S} = \int \underbrace{W(P,\Omega_{\mathrm{R}})\, e^{6N_S^3 N_t \beta P}\ e^{N_S^3 \lambda \cosh\frac{\mu}{T}\Omega_{\mathrm{R}}} \left\langle \cos\left(N_S^3 \lambda \sinh\frac{\mu}{T}\,\Omega_{\mathrm{I}} \right) \right\rangle}_{= F(P,\Omega_{\mathrm{R}})}{}_{P,\Omega_{\mathrm{R}}}\ dP d\Omega_{\mathrm{R}}$$

- $W(P,\Omega_{\mathrm{R}})$ is the provability distribution function in terms of (P,Ω_{R}).
- For the case of $\mu = 0$, when configurations are generated at β and λ, the following equations are satisfied at (P,Ω_{R}) where the configuration generation probability is maximized.

$$\frac{\partial F(P,\Omega_{\mathrm{R}})}{\partial P} = 6N_S^3 N_t \beta + \frac{\partial \ln W(P,\Omega_{\mathrm{R}})}{\partial P} = 0$$

$$\frac{\partial F(P,\Omega_{\mathrm{R}})}{\partial \Omega_{\mathrm{R}}} = N_S^3 \lambda + \frac{\partial \ln W(P,\Omega_{\mathrm{R}})}{\partial \Omega_{\mathrm{R}}} = 0$$

- The peak position of $F(P,\Omega_{\mathrm{R}})$: $\approx \langle P \rangle, \langle \Omega_{\mathrm{R}} \rangle$.
- Right figure: $\langle P \rangle, \langle \Omega_{\mathrm{R}} \rangle$ as functions of β, λ.
- $\frac{\partial \ln W(P,\Omega_{\mathrm{R}})}{\partial P}$ and $\frac{\partial \ln W(P,\Omega_{\mathrm{R}})}{\partial \Omega_{\mathrm{R}}}$ can be measured.



Effect of the complex phase at finite μ

- For finite μ, the peak position is

$$\frac{\partial F(P,\Omega_{\mathrm{R}})}{\partial P} = 6N_S^3 N_t \beta + \frac{\partial \ln W}{\partial P} + \frac{\partial \ln \left\langle \cos\left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)\right\rangle_{P,\Omega_{\mathrm{R}}}}{\partial P} = 0$$

$$\frac{\partial F(P,\Omega_{\mathrm{R}})}{\partial \Omega_{\mathrm{R}}} = N_s^3 \lambda \cosh \frac{\mu}{T} + \frac{\partial \ln W}{\partial \Omega_{\mathrm{R}}} + \frac{\partial \ln \left\langle \cos\left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)\right\rangle_{P,\Omega_{\mathrm{R}}}}{\partial \Omega_{\mathrm{R}}} = 0$$

- Avoiding the sign problem, cumulant expansion:

$$\ln \left\langle \cos\left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)\right\rangle_{P,\Omega_{\mathrm{R}}} = -\frac{1}{2}\left\langle \left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)^2 \right\rangle_c + \frac{1}{4!}\left\langle \left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)^4 \right\rangle_c - \frac{1}{6!}\left\langle \left(N_s^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)^6 \right\rangle_c + \cdots$$

$$\langle \theta^2 \rangle_c = \langle \theta^2 \rangle, \qquad \langle \theta^4 \rangle_c = \langle \theta^4 \rangle - 3\langle \theta^2 \rangle^2, \qquad \langle \theta^6 \rangle_c = \langle \theta^6 \rangle - 15\langle \theta^4 \rangle \langle \theta^2 \rangle + 30\langle \theta^2 \rangle^3, \cdots$$

- We approximate $\ln \left\langle \cos\left(N_S^3 \lambda \sinh \frac{\mu}{T} \Omega_{\mathrm{I}}\right)\right\rangle_{P,\Omega_{\mathrm{R}}} \approx -\frac{1}{2}\left(N_S^3 \lambda \sinh \frac{\mu}{T}\right)^2 \langle \Omega_{\mathrm{I}}^2 \rangle$

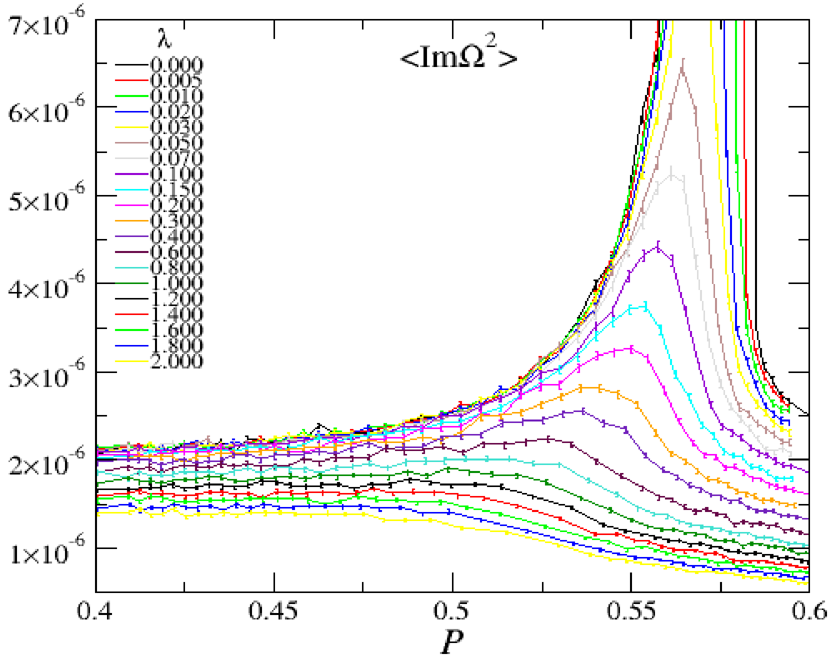
  for qualitative estimation. (Gaussian approximation)

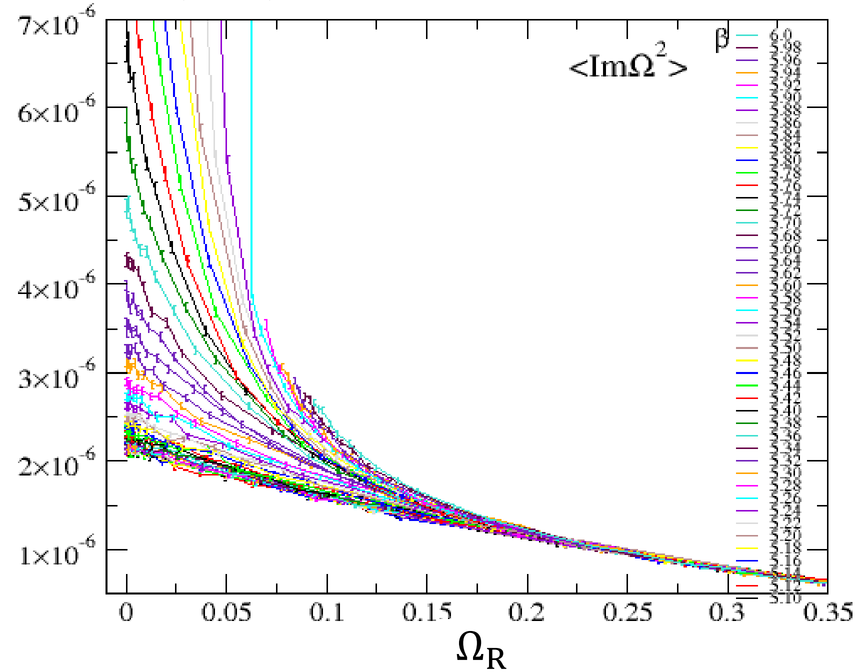
- If we write (β, λ) that are generated by configuration generation at $\mu = 0$ to be $(\langle P \rangle, \langle \Omega_{\mathrm{R}} \rangle)$ as (β_0, λ_0), then (β, λ) that are $(\langle P \rangle, \langle \Omega_{\mathrm{R}} \rangle)$ at finite μ are

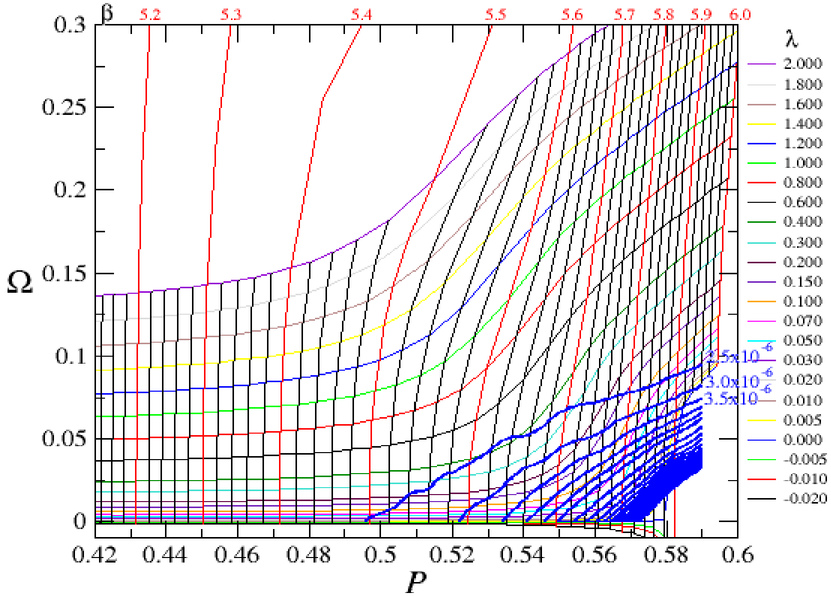
$$\beta = \beta_0 + \frac{N_s^3}{12N_t}\left(\lambda \sinh \frac{\mu}{T}\right)^2 \frac{\partial \langle \Omega_{\mathrm{I}}^2 \rangle_c}{\partial P}, \qquad \lambda \cosh \frac{\mu}{T} = \lambda_0 + \frac{N_s^3}{2}\left(\lambda \sinh \frac{\mu}{T}\right)^2 \frac{\partial \langle \Omega_{\mathrm{I}}^2 \rangle_c}{\partial \Omega_R}$$

Fluctuation of the complex phase

- Variance of Imaginary part of Polyakov loop $\langle \Omega_I^2 \rangle$ Lattice size: $30^3 \times 6$







$$\beta = \beta_0 + \frac{N_s^3}{12 N_t}\left(\lambda \sinh\frac{\mu}{T}\right)^2 \frac{\partial\langle\Omega_I^2\rangle}{\partial P},$$

$$\lambda \cosh\frac{\mu}{T} = \lambda_0 + \frac{N_s^3}{2}\left(\lambda \sinh\frac{\mu}{T}\right)^2 \frac{\partial\langle\Omega_I^2\rangle}{\partial \Omega_R}$$

$\langle \Omega_I^2 \rangle$ is large only near the first-order transition point.

This does not contribute to changing the nature of the phase transition at large μ.

Phase quenched QCD (Ignore complex phase) (isospin μ)

- If we ignore the complex phase, $S_{\text{eff}} = -6N_{\text{site}}\beta^* P - N_s^3\lambda\left(\cosh\frac{\mu}{T}\,\Omega_{\text{R}} \cancel{+ i\sinh\frac{\mu}{T}\,\Omega_{\text{I}}}\right)$
- Simply replace λ at $\mu = 0$ with $\lambda\cosh\frac{\mu}{T}$ to investigate phase quenched QCD.
- Increasing λ means increasing κ, so increasing λ makes the approximation of the hopping parameter expansion worse.
- Therefore, we fix λ at a small value and increase μ.
- If we fix λ, the convergence does not worsen.
- For example, $\lambda = 0.005$

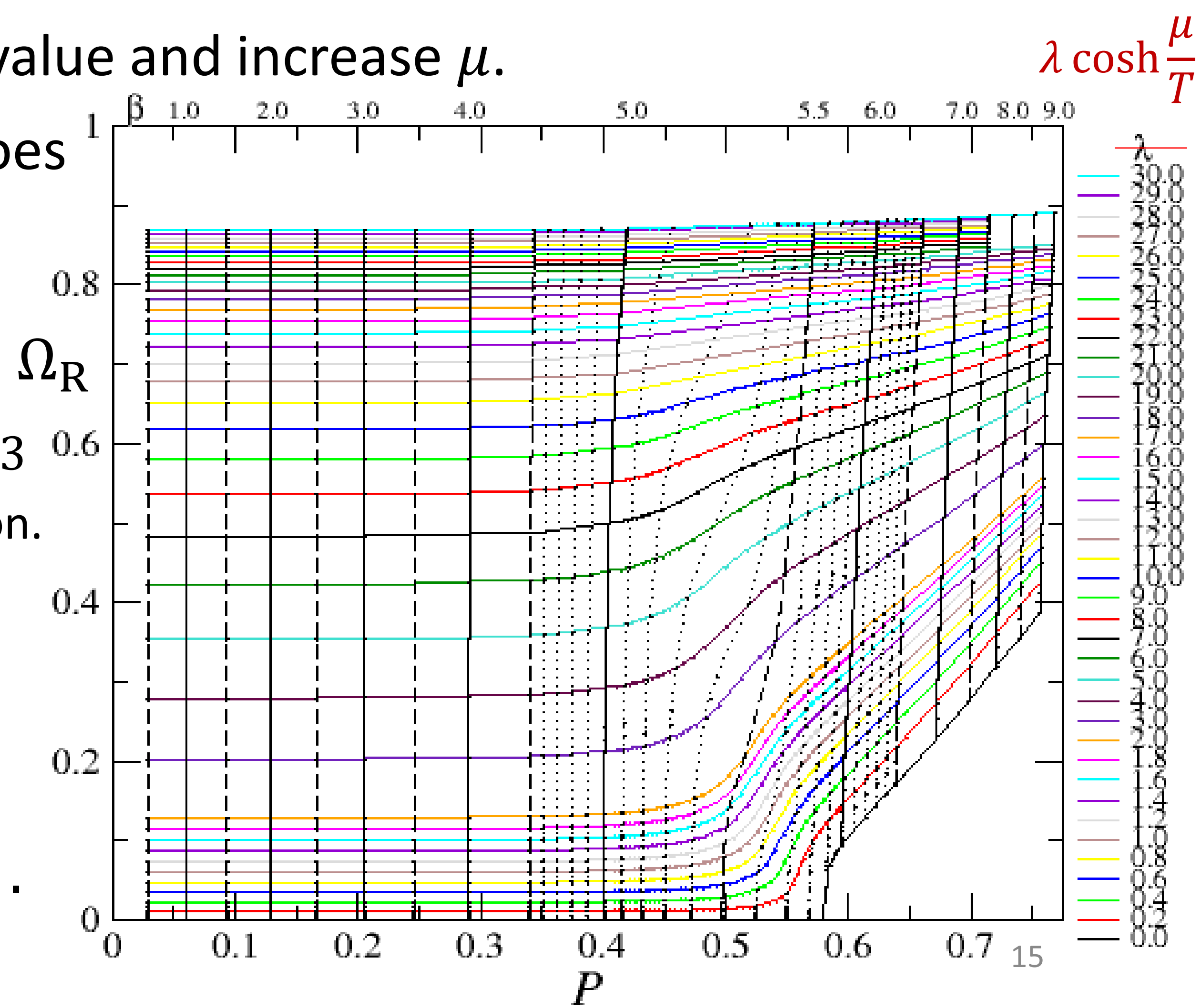
  For $N_{\text{f}} = 2,\ N_t = 6$

  Critical point at $\mu = 0$: $\lambda_{\text{c}} = 0.0013$

  $\lambda = 0.005$ is in the crossover region.

  The convergence of the hopping parameter expansion is good.

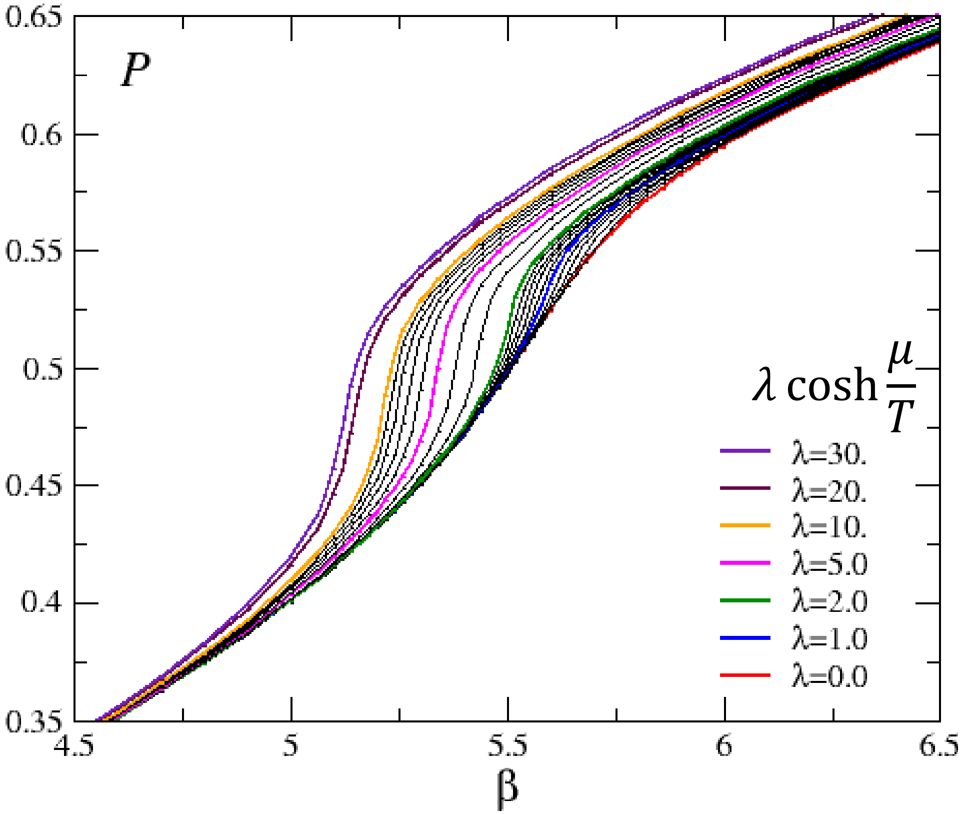
Simulations were performed over a wide range of $\beta, \lambda\cosh\frac{\mu}{T}$.

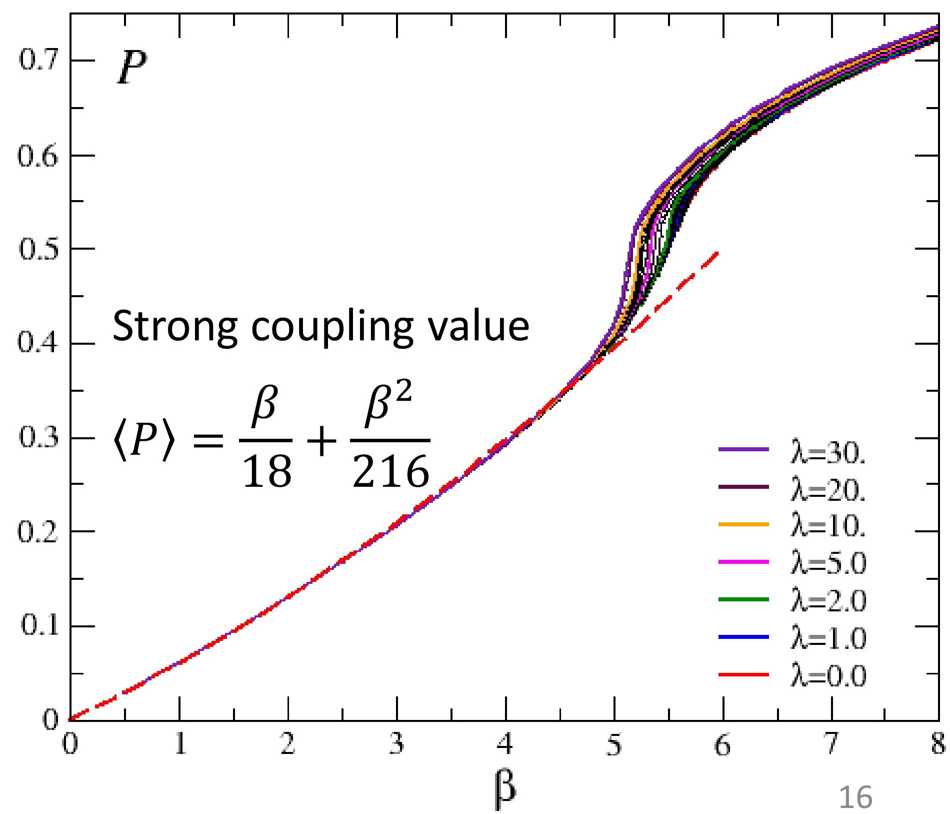


Plaquette in phase quenched QCD (isospin QCD)

- As $\cosh(\mu/T)$ increases, the change in plaquette becomes steeper.
- This behavior is consistent with the existence of a pion-condensed phase at large μ.
- The strong coupling expansion of $\langle P \rangle$ does not depend on $\lambda \cosh(\mu/T)$.
- Confinement phase: $\langle P \rangle$ is consistent with the strong coupling expansion.
- The $\lambda \cosh(\mu/T)$ term forces the deconfinement phase.

Lattice size: $30^3 \times 6$



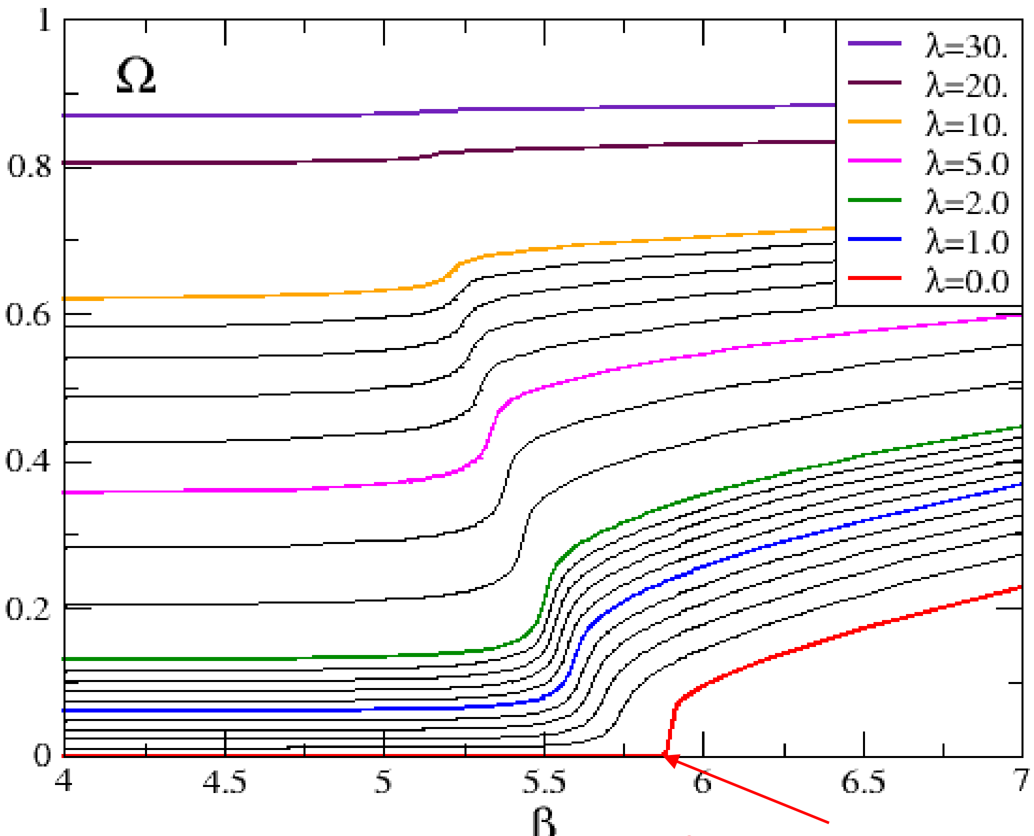


Polyakov loop in phase quenched QCD (isospin QCD)

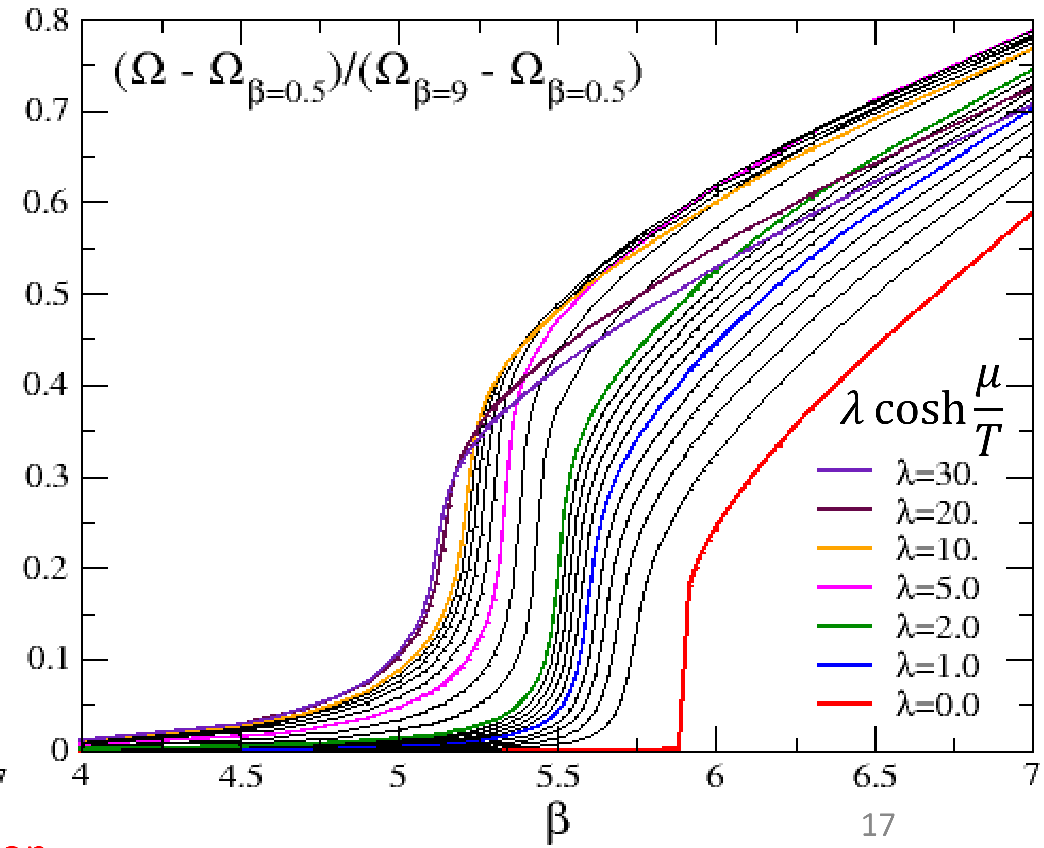
- First order phase transition at $\lambda = 0$
- Regarding the Polyakov loop, if we look closely at the changing part, the change becomes steeper as μ increases.
- $\langle \Omega_{\mathrm{R}} \rangle$ changes almost perpendicular to the horizontal axis
- Shift of β by the complex phase may be important.

$$\frac{\langle \Omega_{\mathrm{R}} \rangle - \langle \Omega_{\mathrm{R}} \rangle_{\beta=0.5}}{\langle \Omega_{\mathrm{R}} \rangle_{\beta=9} - \langle \Omega_{\mathrm{R}} \rangle_{\beta=0.5}}$$

Discontinuity at large μ? → first-order transition?



first-order transition

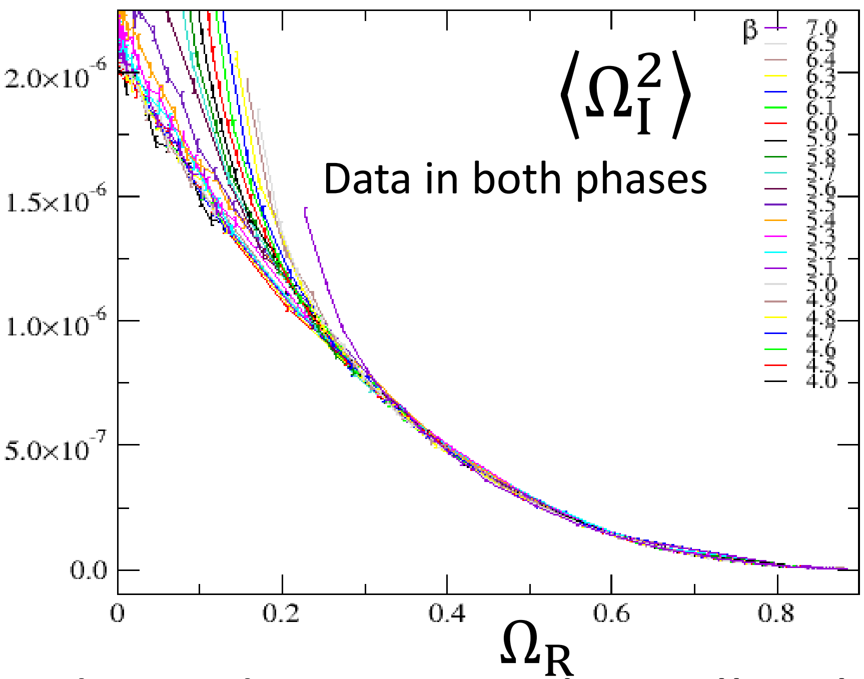


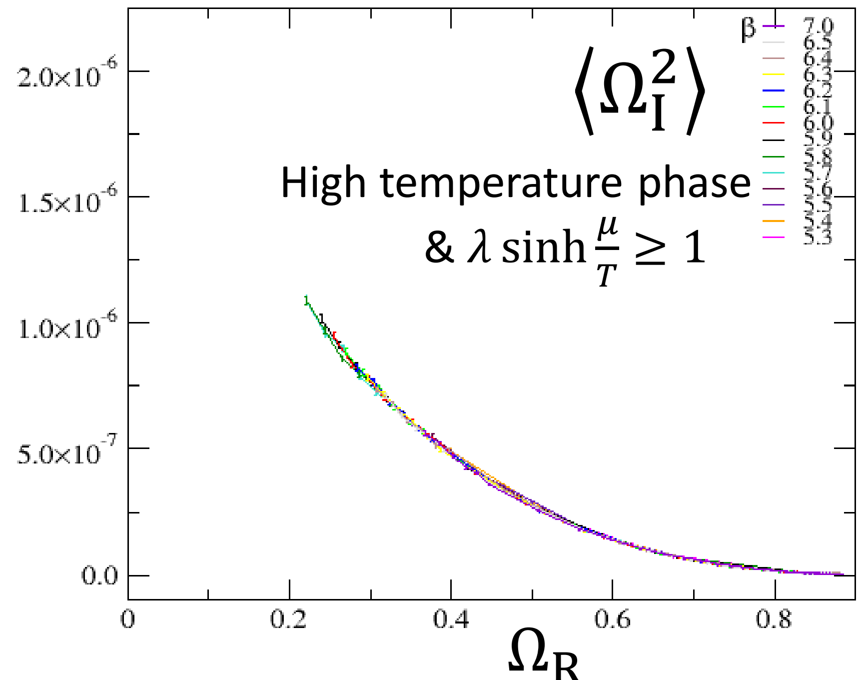
Effect of the complex phase

- We estimate the change in β by the complex phase.

$$\beta = \beta_0 + \frac{N_s^3}{12N_t}\left(\lambda \sinh\frac{\mu}{T}\right)^2 \frac{\partial\langle\Omega_I^2\rangle}{\partial P}$$

Lattice size: $30^3 \times 6$





- P-dependence is much smaller than Ω_R-dependence
- No P- dependence in the high temperature phase: $\left.\frac{\partial\langle\Omega_I^2\rangle}{\partial P}\right|_{\Omega_R} = 0 \quad (\text{High } T)$
- Strong coupling limit (at low temperature)

at $\lambda = 0$: $\langle\Omega_I^2\rangle \approx \frac{1}{2N_s^3N_c^2} + \frac{3}{N_s^3N_c}(P)^{N_t}$

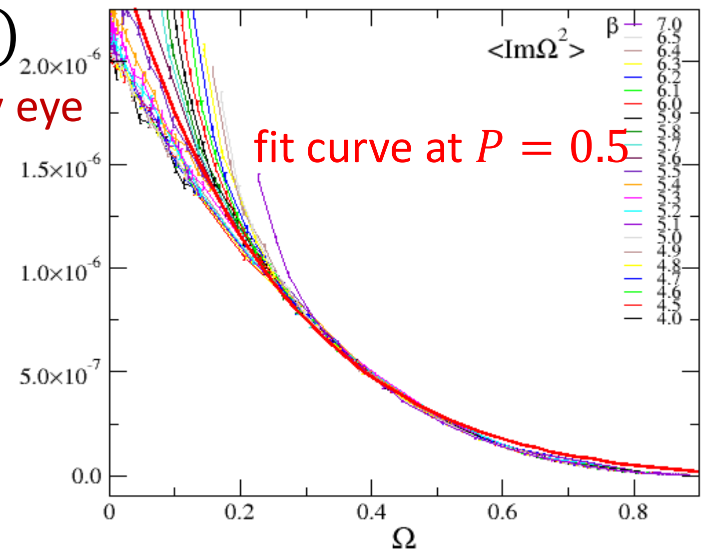
$$\frac{d\langle\Omega_I^2\rangle}{dP} \approx \frac{3N_t}{N_s^3N_c}(P)^{N_t-1} \quad (\text{Low } T)$$

Estimation of β shift by the complex phase

Fit function: $\langle \Omega_{\mathrm{I}}^2 \rangle \approx \underline{\langle \Omega_{\mathrm{I}}^2 \rangle_{\lambda=0}^{\mathrm{strong}}} (1 - \Omega_R) \exp(\underline{-3}\Omega_R)$

Strong coupling limit at $\lambda = 0$

determined by eye

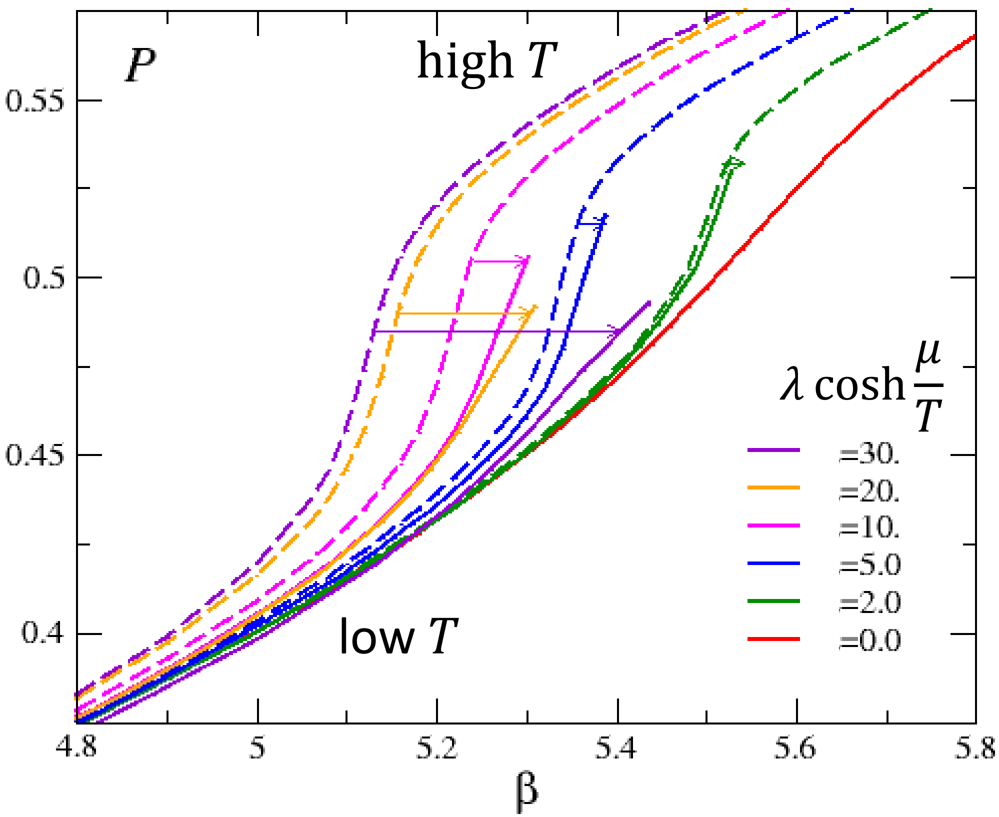


We assume

$$\frac{d\langle \Omega_{\mathrm{I}}^2 \rangle}{dP} = 0 \quad (\text{high } T \text{ phase})$$

$$\frac{d\langle \Omega_{\mathrm{I}}^2 \rangle}{dP} \approx \frac{3N_t}{N_s^3 N_c} (P)^{N_t - 1} (1 - \Omega_R) \exp(-3\Omega_R) \quad (\text{low } T \text{ pahse})$$

$\langle \Omega_{\mathrm{I}}^2 \rangle = 0$ at $\Omega_R = 1$



- β shift: $$\beta = \beta_0 + \frac{N_s^3}{12 N_t} \left(\lambda \sinh \frac{\mu}{T} \right)^2 \frac{\partial \langle \Omega_{\mathrm{I}}^2 \rangle}{\partial P}$$

ex.) $\lambda = 0.005$

- When $\lambda \cosh \frac{\mu}{T} = 5.0$ ($\mu/T = 7.60$) or greater, the graph changes significantly.
- $\lambda \cosh \frac{\mu}{T} = 10, \;\; 20 \rightarrow$ ($\mu/T = 8.29, \;\; 8.99$)
- This suggests the discontinuity in P at the transition point.
- A first-order phase transition is expected at large μ/T.

Summary

- Boundary of first-order transition in the heavy quark region
  - Sign problem is mild near the critical point.
  - As increasing density, the first-order transition region becomes narrower.
- Lee-Yang zeros in the complex μ plane are $Z(\mu)=0$
  - The distance from the real axis is larger than expected.
  - The transition point on the real axis is smaller than μ_R^{LYZ}.
  - If the volume is small, the LY zeros become indistinguishable from the R-W singularity.
- Appearance of first-order phase transitions in the heavy and dense region
  - First, we discussed the nature of the phase transition of phase-quenched finite-density QCD in the heavy quark region.
  - The first-order transition at zero density turns into a crossover as μ is increased, but, when we increase μ further, the change in the plaquette value near the crossover point becomes much steeper.
  - Then, we estimate the effect of the complex phase. The result suggests the appearance of a first-order phase transition region at very large μ.

Strong coupling limit (low temperature phase)

- Strong coupling expansion of $\langle \Omega_{\mathrm{I}}^2 \rangle$ at $\lambda = 0$

$$\langle \Omega_{\mathrm{I}}^2 \rangle = \left\langle \left(\frac{\Omega - \Omega^*}{2i} \right)^2 \right\rangle \approx \frac{1}{2} \langle \Omega \Omega^* \rangle \approx \frac{1}{2N_s^6 N_c^2} \sum_{x,y} \left\langle \mathrm{tr}(UU \cdots U)_x \mathrm{tr}\left(U^\dagger U^\dagger \cdots U^\dagger\right)_y \right\rangle$$

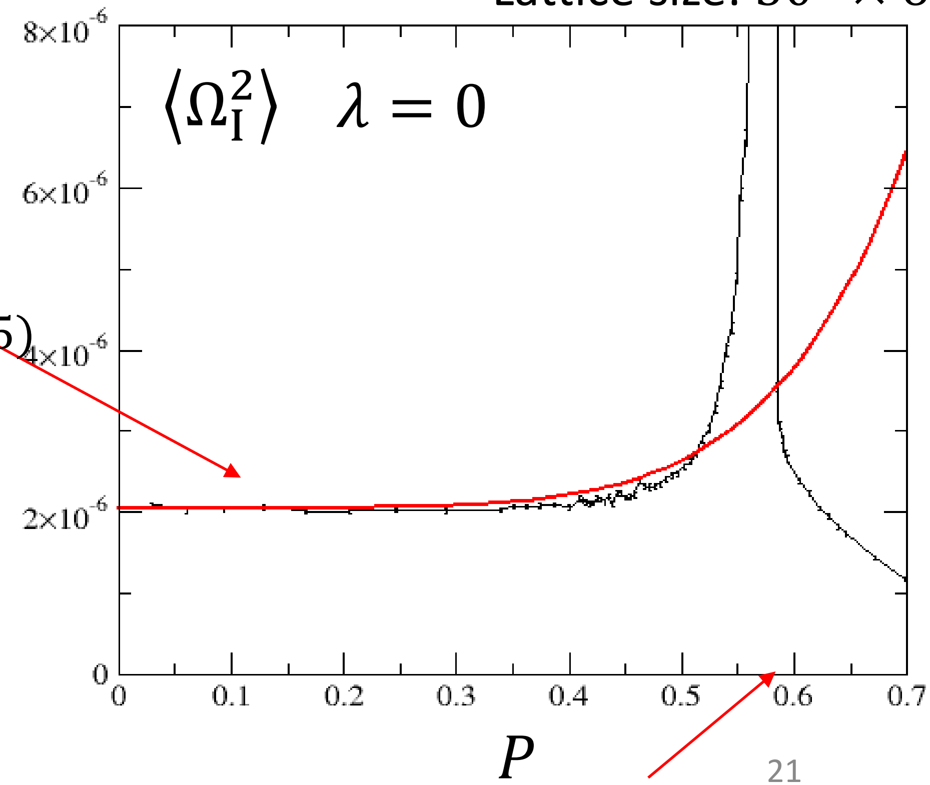
$$\approx \frac{N_s^3}{2N_s^6 N_c^2} \left\langle \mathrm{tr}(UU \cdots U)_x \mathrm{tr}\left(U^\dagger U^\dagger \cdots U^\dagger\right)_x \right\rangle + \frac{6N_s^3}{2N_s^6 N_c^2} \left\langle \mathrm{tr}(UU \cdots U)_x \mathrm{tr}\left(U^\dagger U^\dagger \cdots U^\dagger\right)_{x+1} \right\rangle + \cdots$$

$$\approx \frac{N_s^3}{2N_s^6 N_c^2} + \frac{6N_s^3}{2N_s^6 N_c^2} N_c \left(\frac{\beta}{2N_c^2} \right)^{N_t} = \frac{1}{2N_s^3 N_c^2} + \frac{3}{N_s^3 N_c} \left(\frac{\beta}{2N_c^2} \right)^{N_t}$$

- Plaquette is given by $P \approx \dfrac{\beta}{2N_c^2}$
- Thus, $\langle \Omega_{\mathrm{I}}^2 \rangle \approx \dfrac{1}{2N_s^3 N_c^2} + \dfrac{3}{N_s^3 N_c} (P)^{N_t}$
- consistent with the simulation data $(P < 0.5)$
- but P-dependence is very small.
- A large increase near the transition point.
- The derivative:

$$\frac{d\langle \Omega_{\mathrm{I}}^2 \rangle}{dP} \approx \frac{3N_t}{N_s^3 N_c} (P)^{N_t - 1}$$

Lattice size: $30^3 \times 6$



First-order phase transition

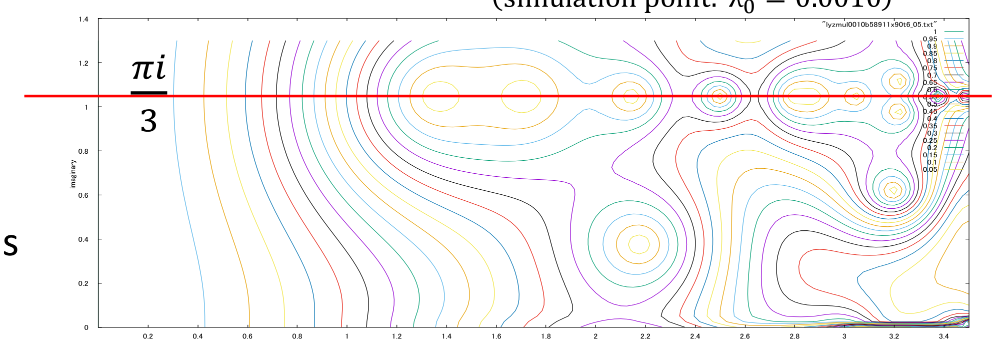
Roberge-Weiss singularity

- Z_3 center symmetry $\qquad U_{\text{center}} = \omega I, \qquad \omega = \{1, e^{\frac{2\pi i}{3}}, e^{\frac{4\pi i}{3}}\}$
- Gauge action and integral measure: invariant, Polyakov loop: $\langle\Omega\rangle \Rightarrow \omega\langle\Omega\rangle$
- Quark determinant: $\det M\,(\kappa,\mu) \Rightarrow \det M\left(\kappa, \mu + \frac{2\pi i}{3}\right)$
    - For Full QCD, Z_3 center symmetry: broken. But, symmetric if one changes μ.
- Roberge-Weiss periodicity: $Z(\beta,\lambda^*,\mu) = Z\left(\beta,\lambda^*,\mu - \frac{2\pi i}{3}\right) = Z\left(\beta,\lambda^*,\mu + \frac{2\pi i}{3}\right)$
- R-W singularity: branch cut line at $\mathrm{Im}\mu = \pm\frac{\pi}{3}$ in the complex μ plane
- ex.) $\mu_R = 0$, for the case of heavy quark effective theory,

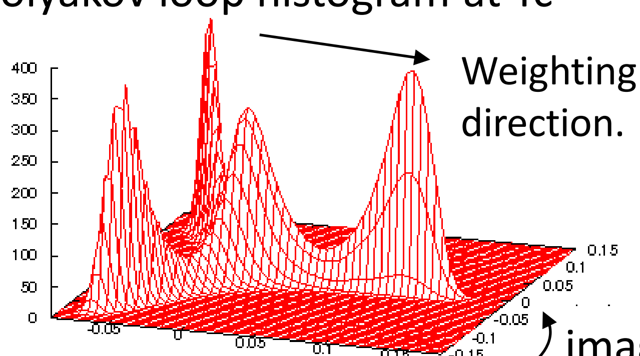
$$S_{\text{eff}} = -6N_{\text{site}}\beta^* P - N_s^3\lambda^*\mathrm{Re}\left(\mathrm{e}^{i\mu_I/T}\Omega\right)$$

$N_t = 6, N_s = 90, \beta^* = 5.8911, \lambda^* = 0.0005$

(simulation point: $\lambda_0^* = 0.0010$)



Polyakov loop histogram at Tc



Weighting in the positive real axis direction. μ_I/T rotates Ω.

imaginary

real

At $\mu_I/T = \pm\frac{\pi}{3}$, the peak with the highest weight is swapped.

$\Rightarrow$ Branch cut (If one peak in the low-T phase, there is no cut.)

Lee-Yang zero and Histogram of $\Omega_R = Re\Omega$

- Indicator of Lee-Yang zero: $Z_{\rm nor}(\beta,\lambda) = \frac{|Z(\beta,\lambda)|}{Z(\beta,\lambda_R)}$, (complex $\lambda = \lambda_R + i\lambda_I$)

$$S_{\rm eff} = -6N_{\rm site}\beta P - N_s^3\lambda\Omega_{\rm R}$$

$$Z_{\rm nor}(\beta,\lambda) = \frac{1}{Z}\left|\int DU\, e^{-{\rm Re}S} e^{iN_s^3\lambda_I\Omega_{\rm R}}\right| = \left|\left\langle e^{iN_s^3\lambda_I\Omega_{\rm R}}\right\rangle_{(\beta,\lambda_R)}\right| = \left|\int e^{iN_s^3\lambda_I\Omega_{\rm R}} p_\Omega d\Omega_{\rm R}\right|$$

  - $p_\Omega(\Omega_{\rm R})$: Histogram of $\Omega_{\rm R}$. Its Fourier transform is $Z_{\rm nor}(N_s^3\lambda_I)$
- In the case of a strong first-order phase transition, Ejiri, Phys.Rev.D73, 054502(2006)
  - In the complex plane, they lie on a straight line perpendicular to the real axis.
  - $p_\Omega(\Omega_R)$ has two peaks and the distance between the peaks is $\Delta\Omega$.
  - If $N_s^3\lambda_I\Delta\Omega \approx \pi + 2\pi n, (n = 0,1,2,\cdots)$, then $Z(\beta,\lambda) = 0$.
- Scaling law near the critical point (spatial length: $L = N_s a$)
  - Scaling Function $f(tL^{y_t}, hL^{y_h})$ (temperature: $t = (T - T_c)/T_c$, magnetic field: h)
  - $\Omega_{\rm R}$ corresponds to magnetization. Near the critical point, $\Delta\Omega = 2(\Omega_{\rm R} - \langle\Omega_{\rm R}\rangle) \sim L^{-3+y_h}$
  - Scaling law for the histogram of $\Omega_{\rm R}$: $p_\Omega(\Delta\Omega L^{3-y_h};\ tL^{y_t}, hL^{y_h})$

Kiyohara et al., Phys.Rev.D104, 114509(2021)

- Scaling law for $Z_{\rm nor}(\beta,\lambda)$
  - Conjugate variable of $\Omega_{\rm R}L^{3-y_h}$ in the Fourier transform: $N_s^3\lambda_I/L^{3-y_h} = \lambda_I L^{y_h}a^{-3}$
  - Temperature variable: $t \sim \lambda - \lambda_c$, $(h = 0)$. $Z_{\rm nor}$ is a function of $L^{y_h}\lambda_I$.
  - Lee-Yang zero behaves as $\lambda_I \sim L^{-y_h}$

Finite volume scaling analysis of histogram

$N_t = 4$

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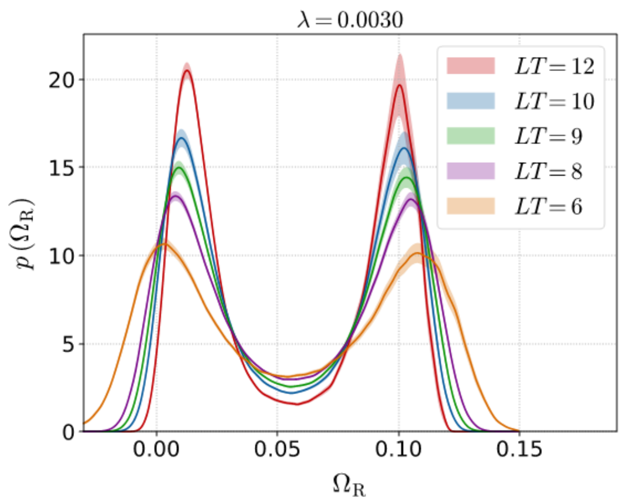
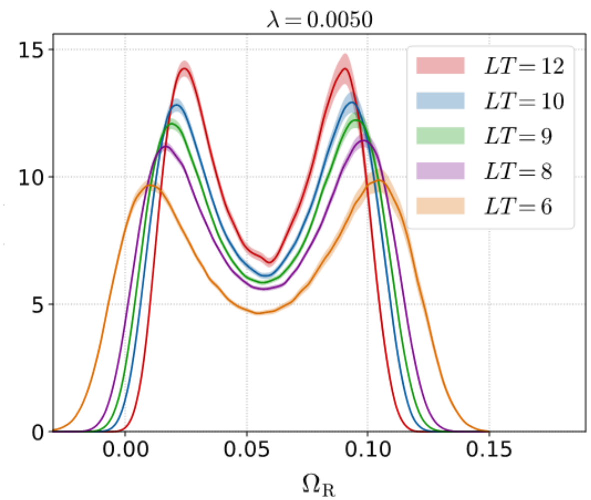
- Histogram of Polyakov loop at transition point $p_\Omega(\Omega_R)$ $LT = N_s/N_t$

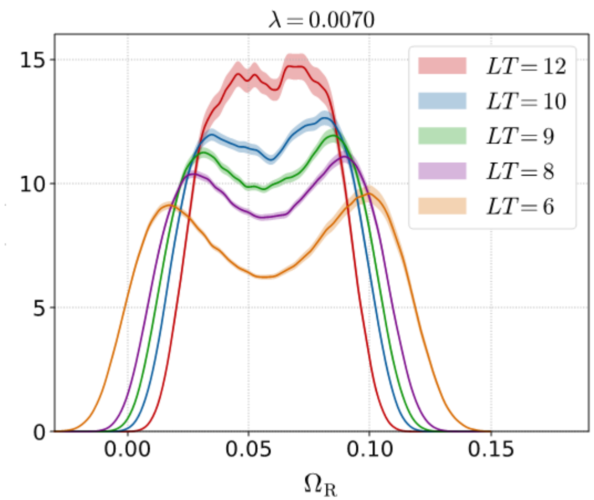
First-order transition



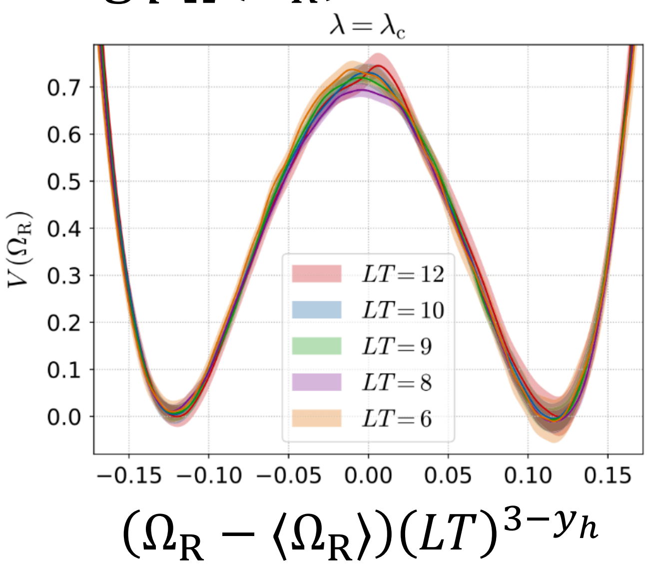
Near the critical point



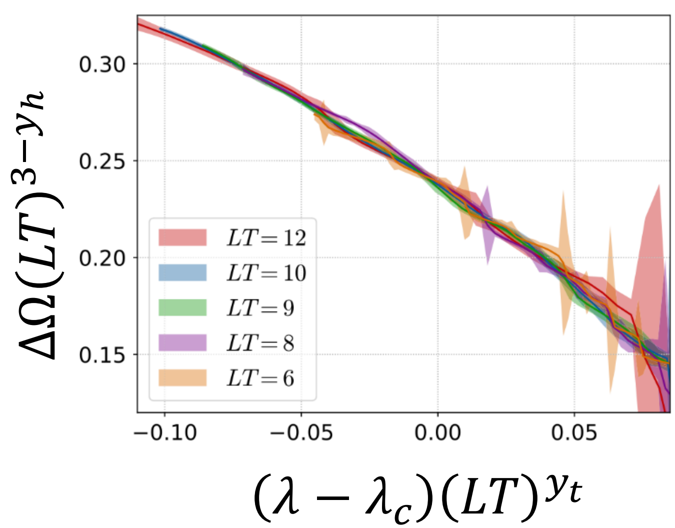
Crossover



- $-\log p_\Omega(\Omega_R)$ at the critical point



Peak Gap $\Delta\Omega$ vs $\lambda - \lambda_c$



Histogram of $(\Omega_R - \langle\Omega_R\rangle)(LT)^{3-y_h}$

Temperature variable: decreases the gap between the two peaks.

$$tL^{y_t} \sim (\lambda - \lambda_c)(LT)^{y_t}$$

At the phase transition point, the magnetic field variable is h=0. (Changing the height difference between the two peaks)

Ising universality class: $y_h = 2.482$, $y_t = 1/\nu = 1/0.630$