Lattice computation of inclusive processes

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Shoji Hashimoto（KEK, SOKENDAI）

SOKENDAI

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Inclusive processes = sum over final states

Conveniently written using the spectral function

$$R(X) = \int_0^\infty d\omega\, S(\omega; X)\rho(\omega) \qquad \text{with} \qquad \rho(\omega) \propto \sum_X \delta(\omega - E_X)|\langle X|J|0\rangle|^2$$

With a vacuum:

- HVP for muon g-2
- Hadronic tau decays

Can be viewed as a "**smeared spectrum**":

Easily reconstructed, once the spectral function ρ(ω) is obtained.
Often not the case, then what to do?

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With a vacuum:

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With a specific initial state:

- Semi-leptonic decays
- Inclusive l-N cross sections

Can be viewed as a "**smeared spectrum**":

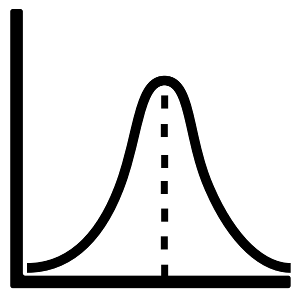
Easily reconstructed, once the spectral function ρ(ω) is obtained.
Often not the case, then what to do?

Lattice correlator as a smeared spectrum

all possible states contribute

Lattice correlator: $$C(t) = \int_0^\infty d\omega\, \rho(\omega) e^{-\omega t} \qquad \sim \langle 0 | J\, e^{-\hat{H} t}\, J | 0 \rangle$$

Smeared spectrum: $$R(X) = \int_0^\infty d\omega\, S(\omega; X) \rho(\omega) \qquad \sim \langle 0 | J S(\hat{H}; X) J | 0 \rangle$$



δ-function

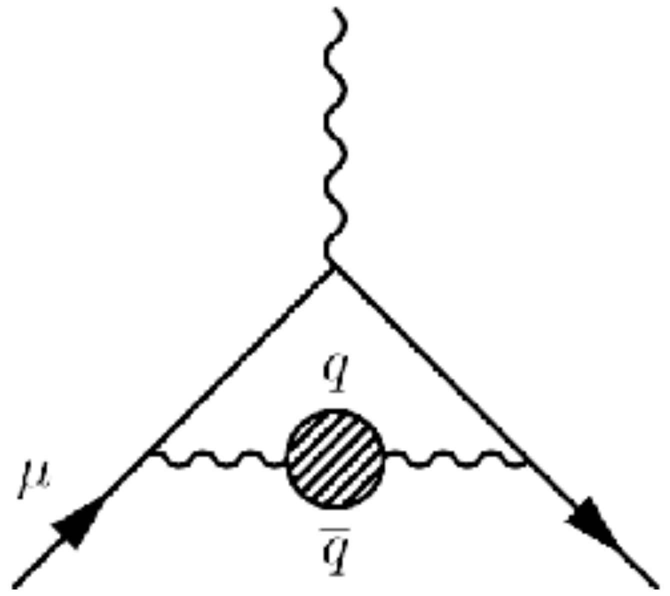
One can consider **any** function.

c.f. spectral func: $$\rho(\omega) \propto \sum_X \delta(\omega - E_X) |\langle X | J | 0 \rangle|^2 \qquad \sim \langle 0 | J\, \delta(\omega - \hat{H})\, J | 0 \rangle$$

HVP contrib to muon g-2

$$a_\mu^{\mathrm{HVP,LO}} \sim \int_0^\infty \frac{d\omega}{\omega^3} K(\omega^2) \cdot \omega^2 \rho(\omega)$$

slowly varying func

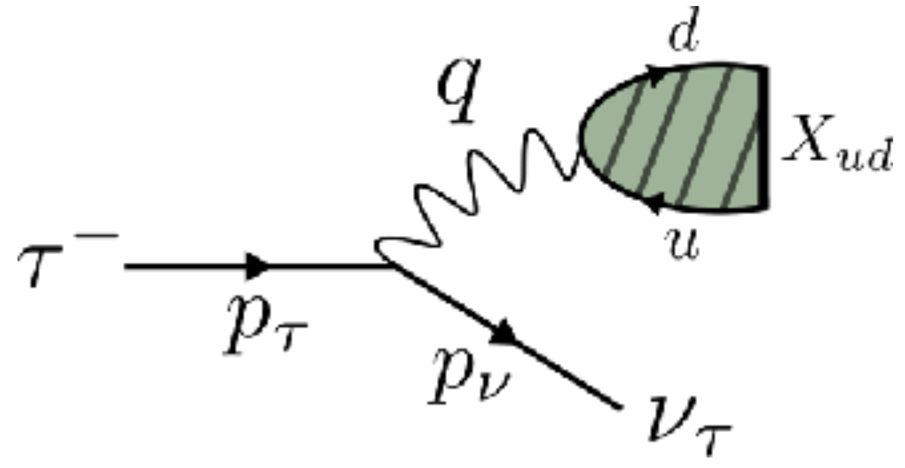


Time-momentum representation (Bernecker-Meyer, **Mainz**!) is available for space-like quantities.

Hadronic τ decay

c.f. work by ETMC

$$\Gamma^{(\tau)} \sim \int_0^{m_\tau} \frac{d\omega}{\omega^3} \left(1 - \frac{\omega^2}{m_\tau^2}\right) \left(1 + 2\frac{\omega^2}{m_\tau^2}\right) \cdot \omega^2 \rho_T(\omega)$$



heavier weight on the low-energy end ~ $2m_\pi$

Alternative path

when time-momentum rep is unavailable (time-like)

Compute $\int_0^\infty d\omega\, K(\omega)\rho(\omega)$ from $C(t) = \int_0^\infty d\omega\, \rho(\omega) e^{-\omega t}$

$\sim \langle 0|J K(\hat{H}) J|0\rangle$ $\sim \langle 0|J\, e^{-\hat{H}t}\, J|0\rangle$

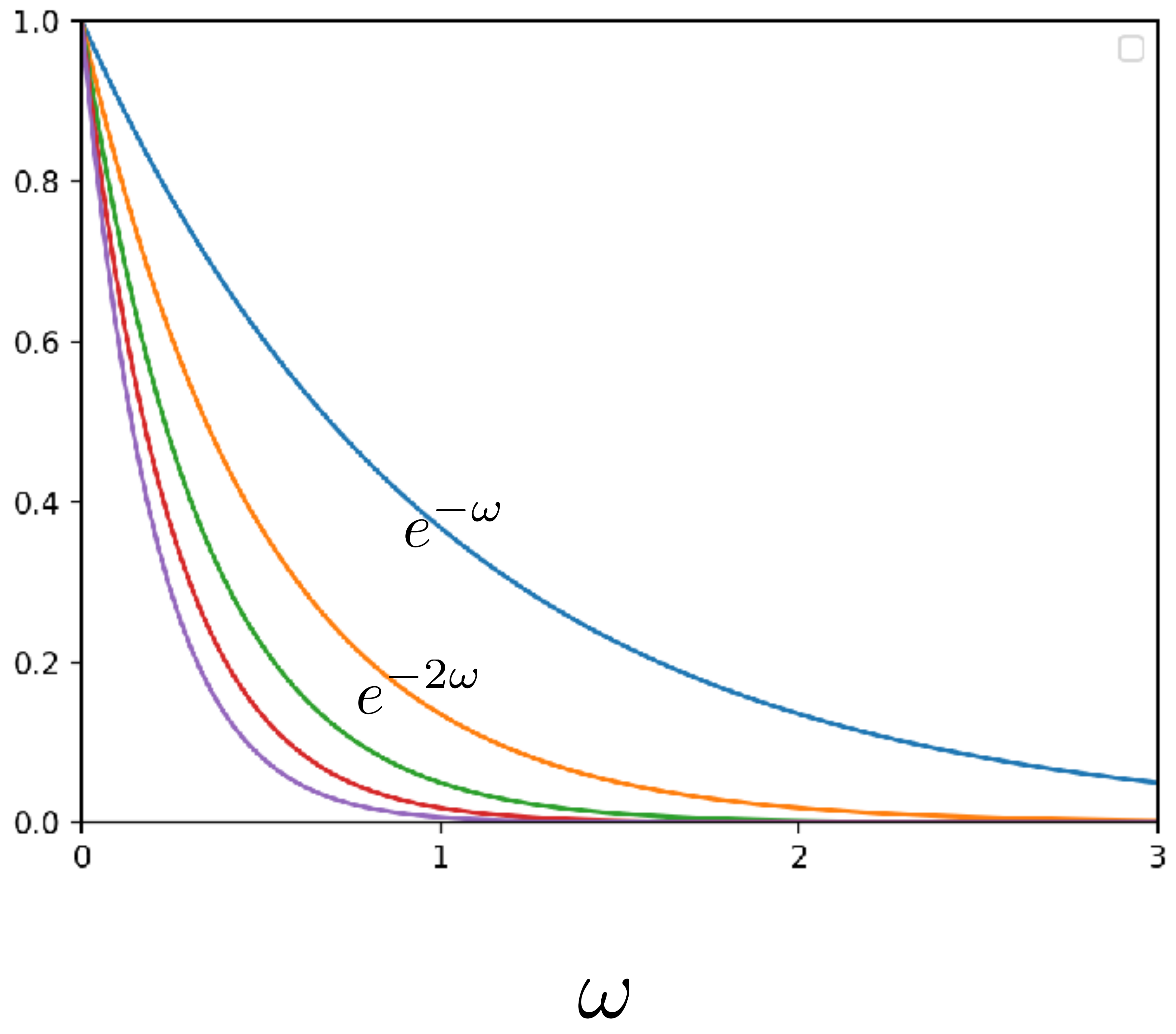
= **Establish an approximation**

$$K(\hat{H}) \simeq k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-N\hat{H}}$$

combination of $C(t)$ with various t;

Approximation?

$$K(\hat{H}) \simeq k_0 + k_1 e^{-\hat{H}} + k_2 e^{-2\hat{H}} + \cdots + k_N e^{-N\hat{H}}$$



- Not always possible; when the function varies rapidly, in particular.
- Some available methods
  - Modified Backus-Gilbert

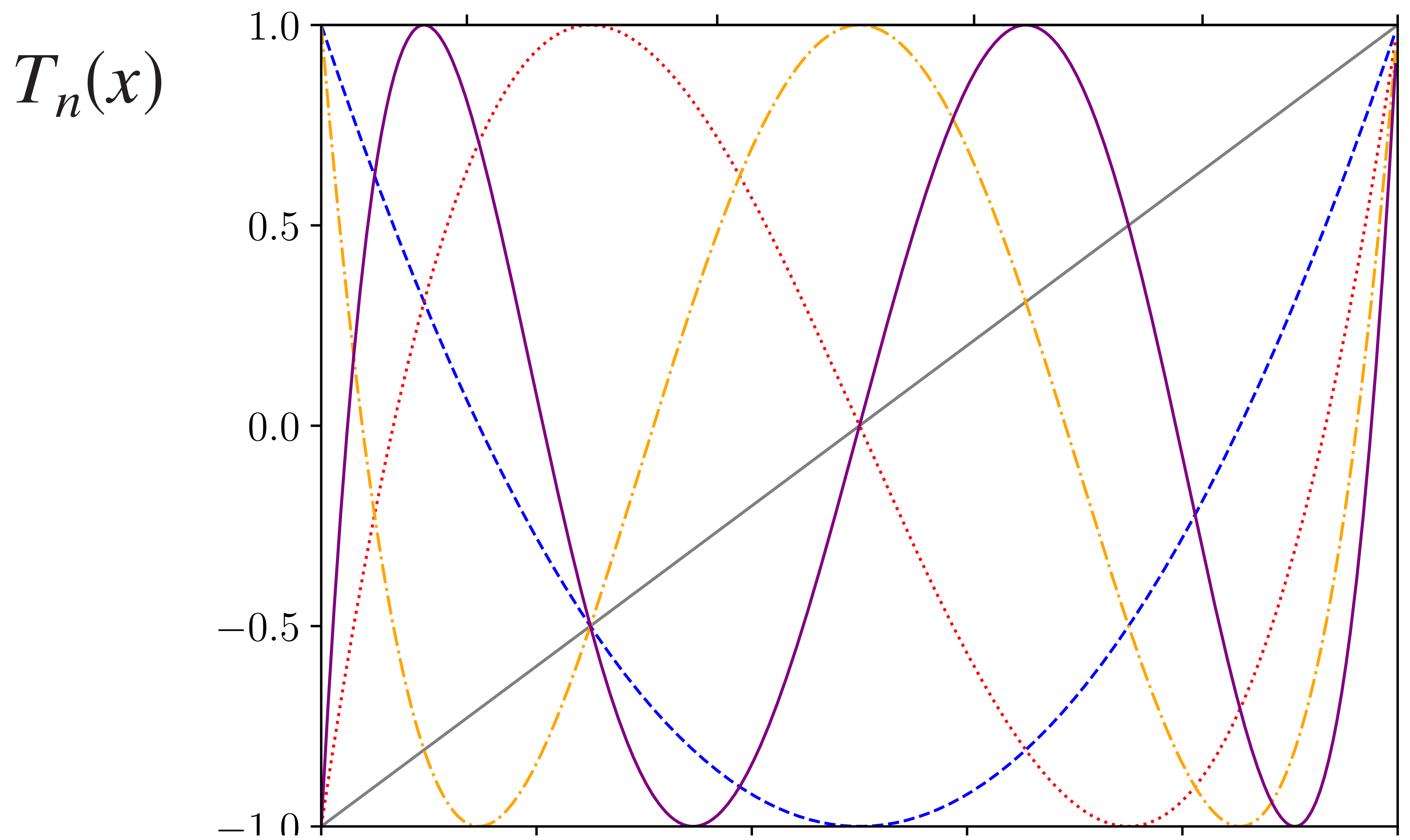
    Hansen, Lupo, Tantalo, arXiv:1903.06476
  - Chebyshev polynomial

    Bailas, Ishikawa, SH, arXiv:2001.11779

Systematic errors?

Chebyshev polynomials approximation

$$f(x) \simeq \frac{c_0}{2} + \sum_{j=1}^{N-1} c_j T_j(x) \quad ; \quad x = e^{-\omega}$$



- Expansion in terms of orthogonal polynomials
  - *Lanczos* (1952): convergence of Chebyshev is the fastest among other orthogonal polynomials.
    Mostly the case; there are exceptions, though.
- Coefficients c_j easily obtained to arbitrary N
  How fast is the **convergence**? See below
- Each term satisfies $|T_j(x)| \leqq 1$. Upper limit of the ignored terms (truncation errors) is known
- Typical resolution at O(N) ~ $1/N$
  - thus, the energy resolution ~ $1/N$
- Each j from $C(j = t)$; precise approx needs large time sep

try to approx a "delta" function with N = 20

$$f(x) \simeq \frac{c_0}{2} + \sum_{j=1}^{N-1} c_j T_j(x) \quad ; \quad x = e^{-\omega}$$

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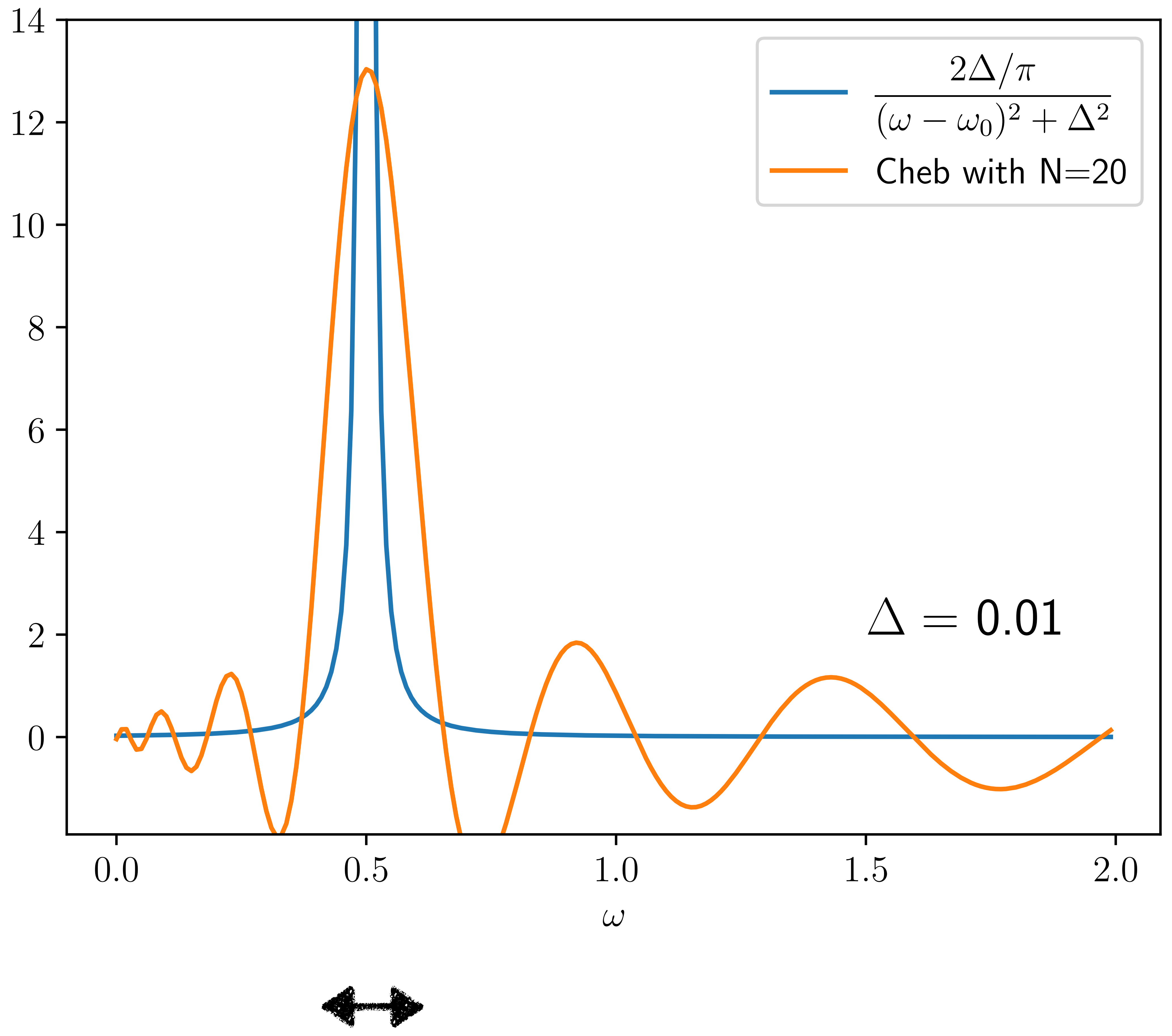

Resolution is about 1/N or a bit larger.

$$f(x) \simeq \frac{c_0}{2} + \sum_{j=1}^{N-1} c_j T_j(x) \quad ; \quad x = e^{-\omega}$$

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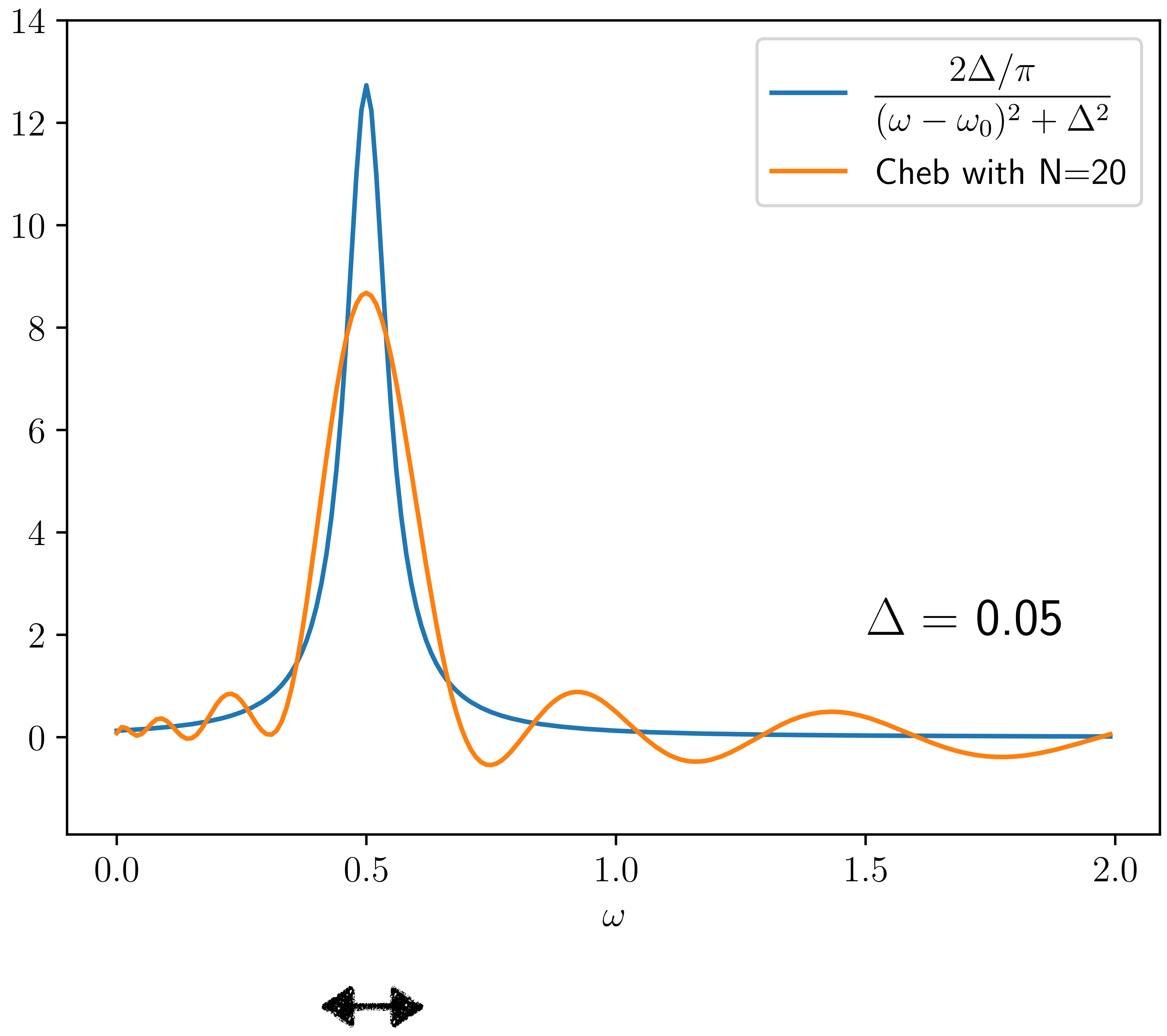

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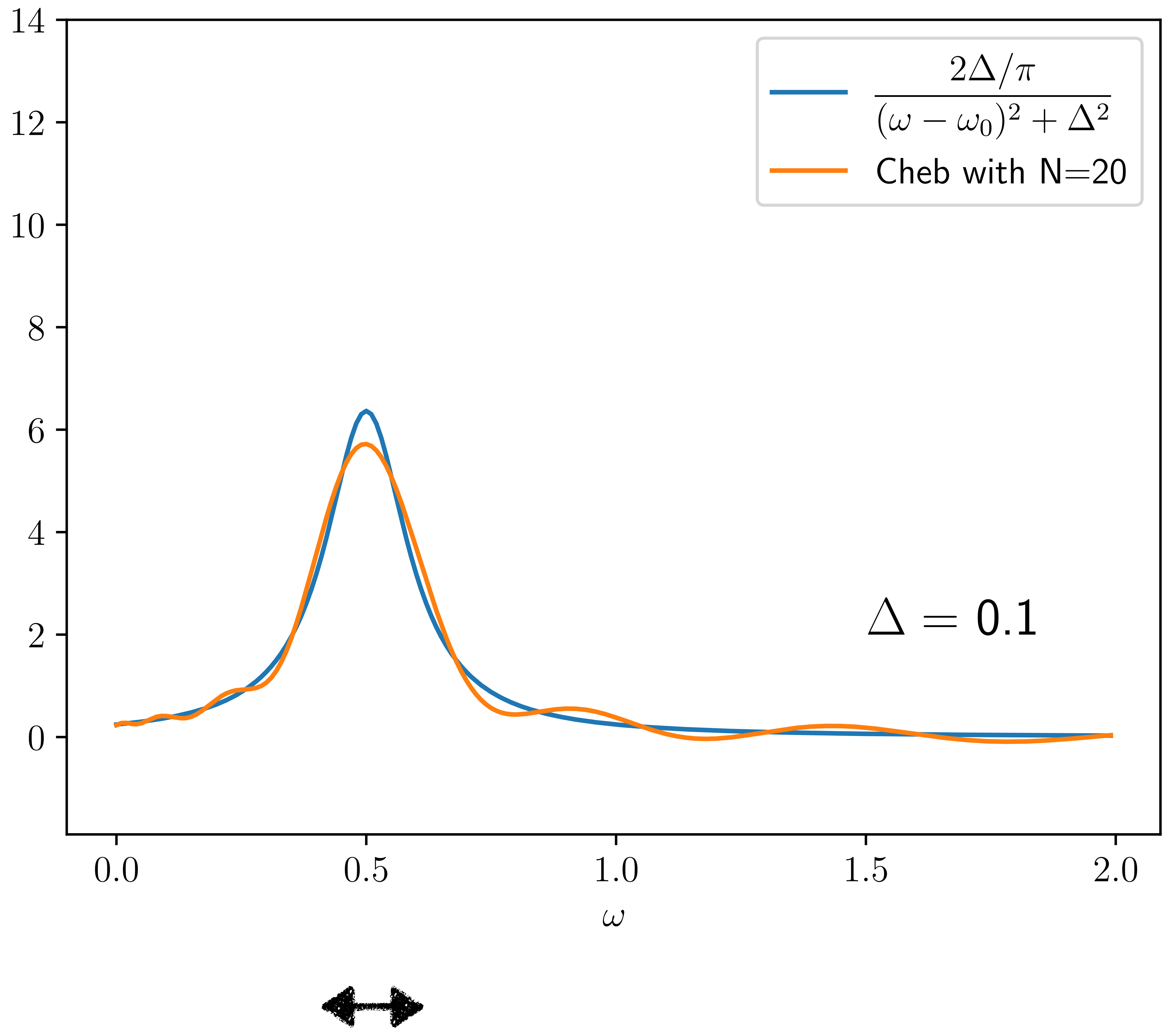

- Typical resolution at O(N) ~ 1/N
  - thus, the energy resolution ~ 1/N (Look at the nodes)
- Independent of the target function
  - Limit of $\Delta \rightarrow 0$ has to be taken with (or after) $N \rightarrow \infty$

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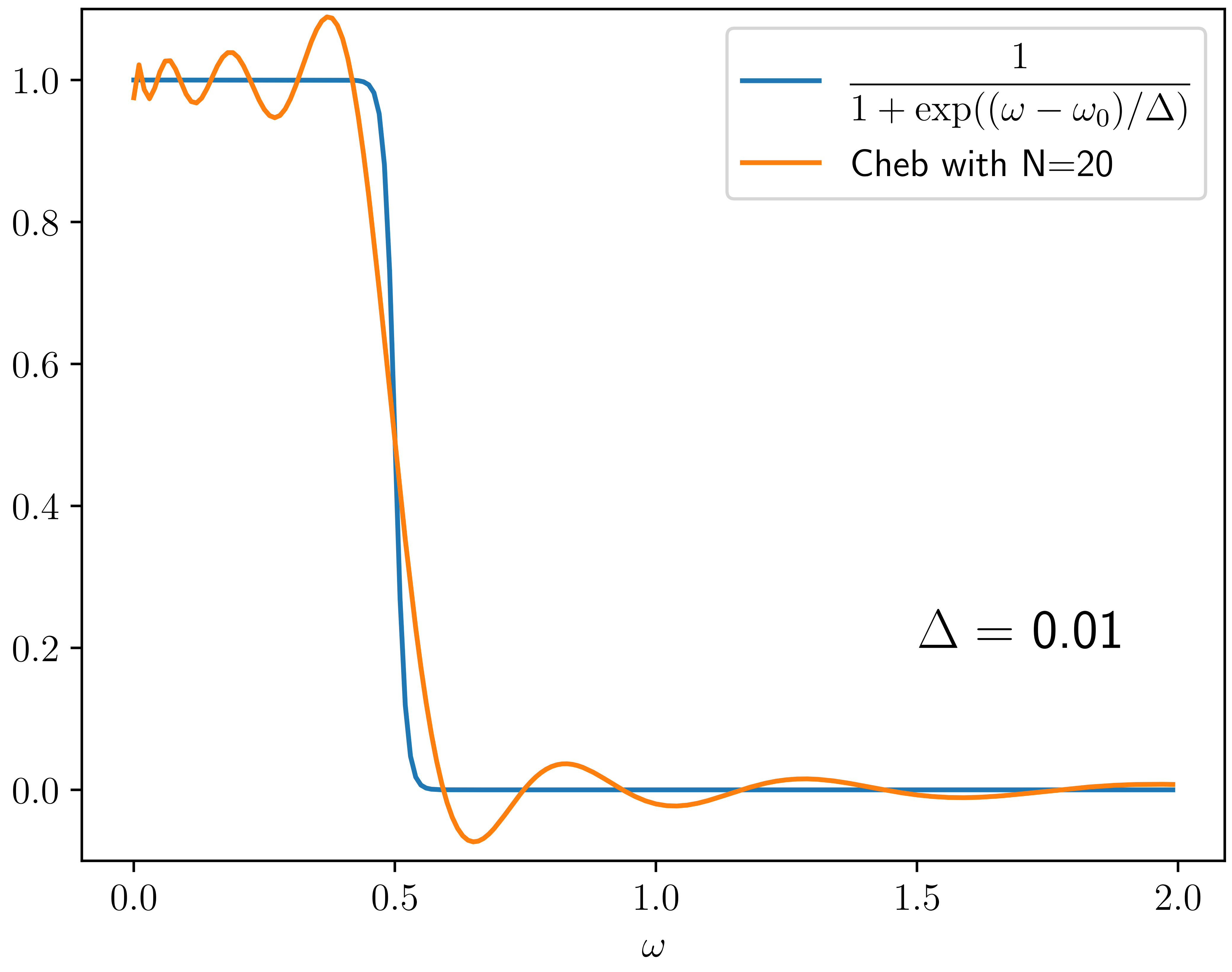
try to approx a “step” function with *N* = 20



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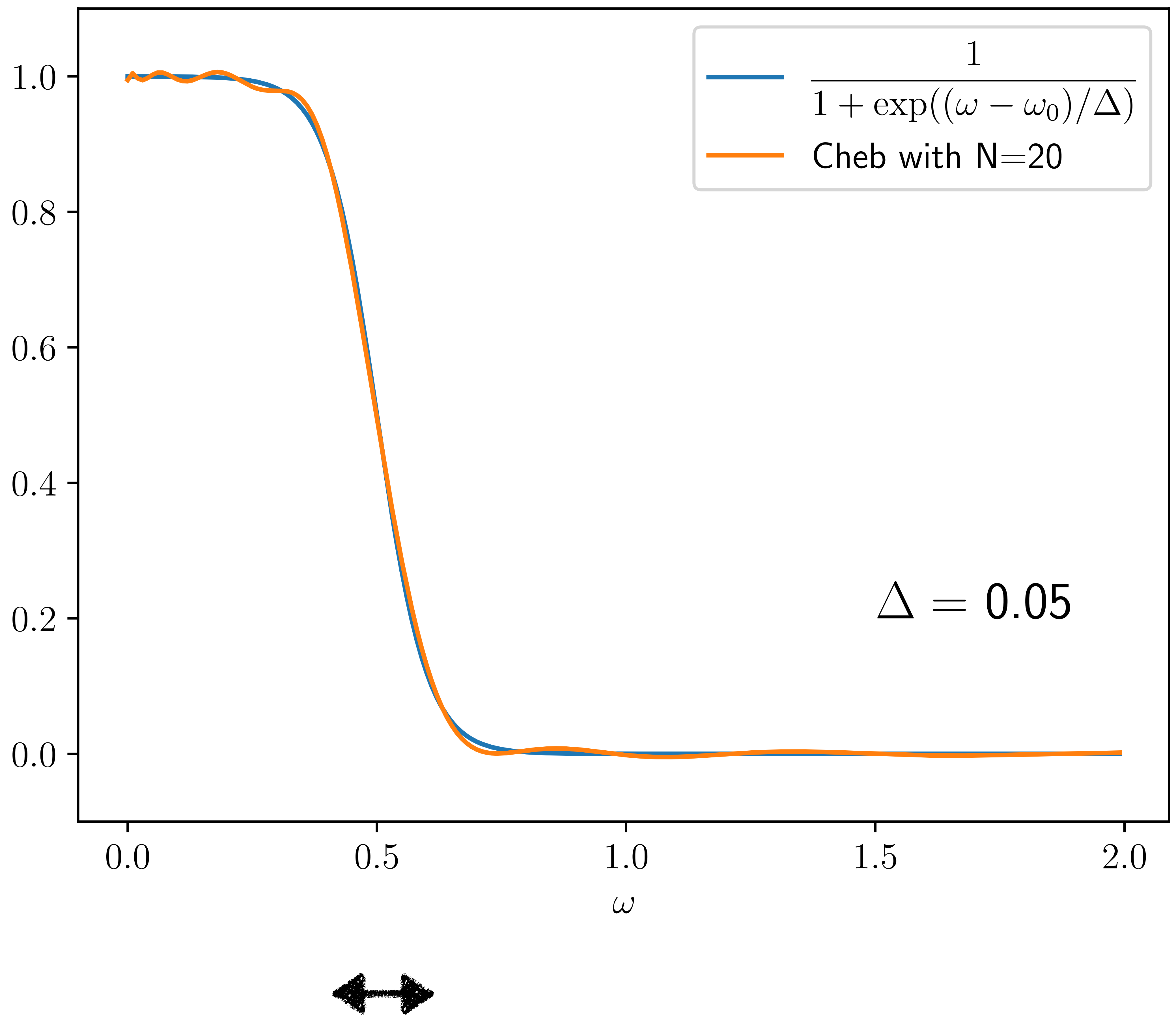

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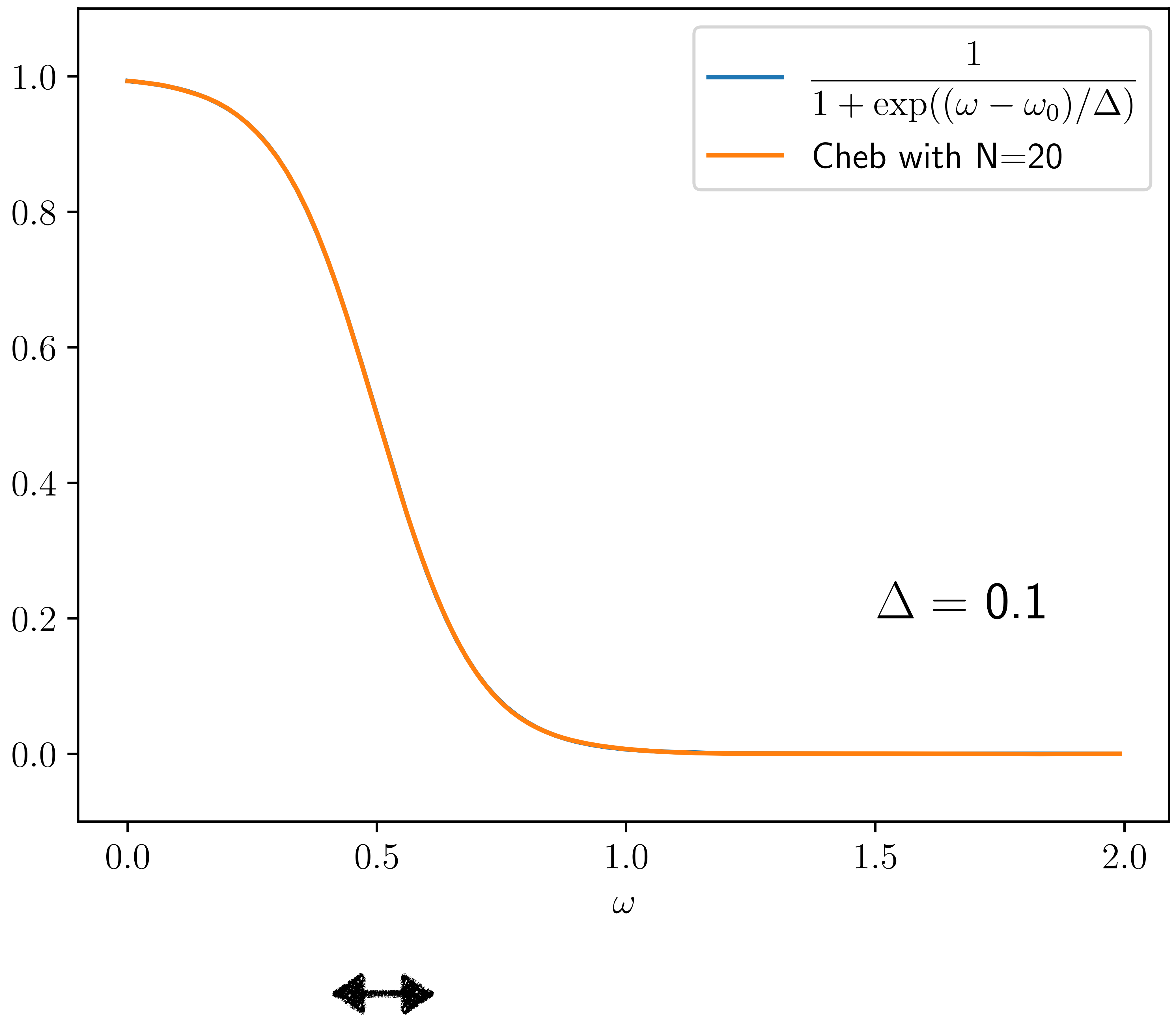
Or, restrict the application to sufficiently smooth functions.

try to approx a "step" function with *N* = 20

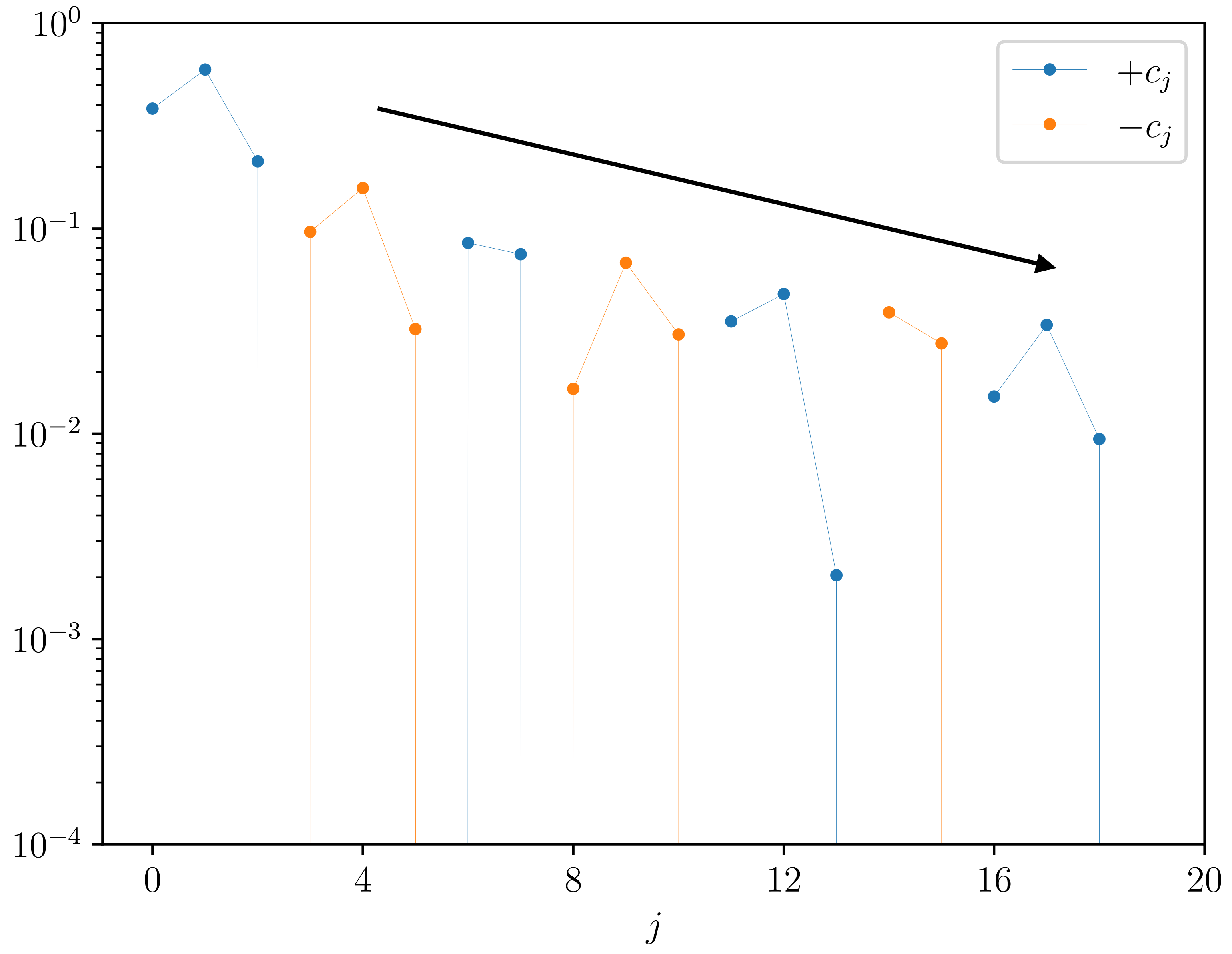


Resolution is about 1/N or a bit larger.

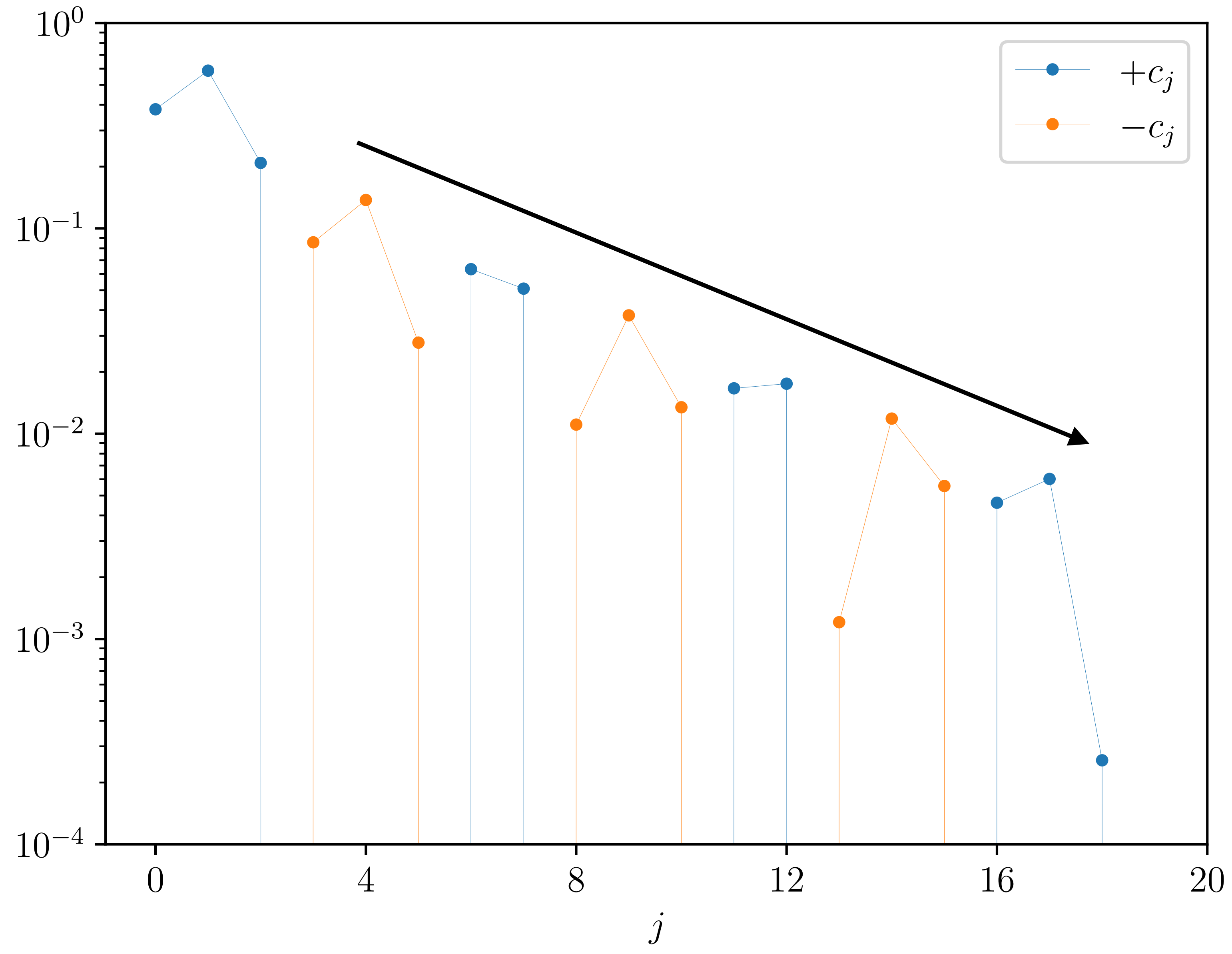
Convergence of the expansion

coefficients c_j

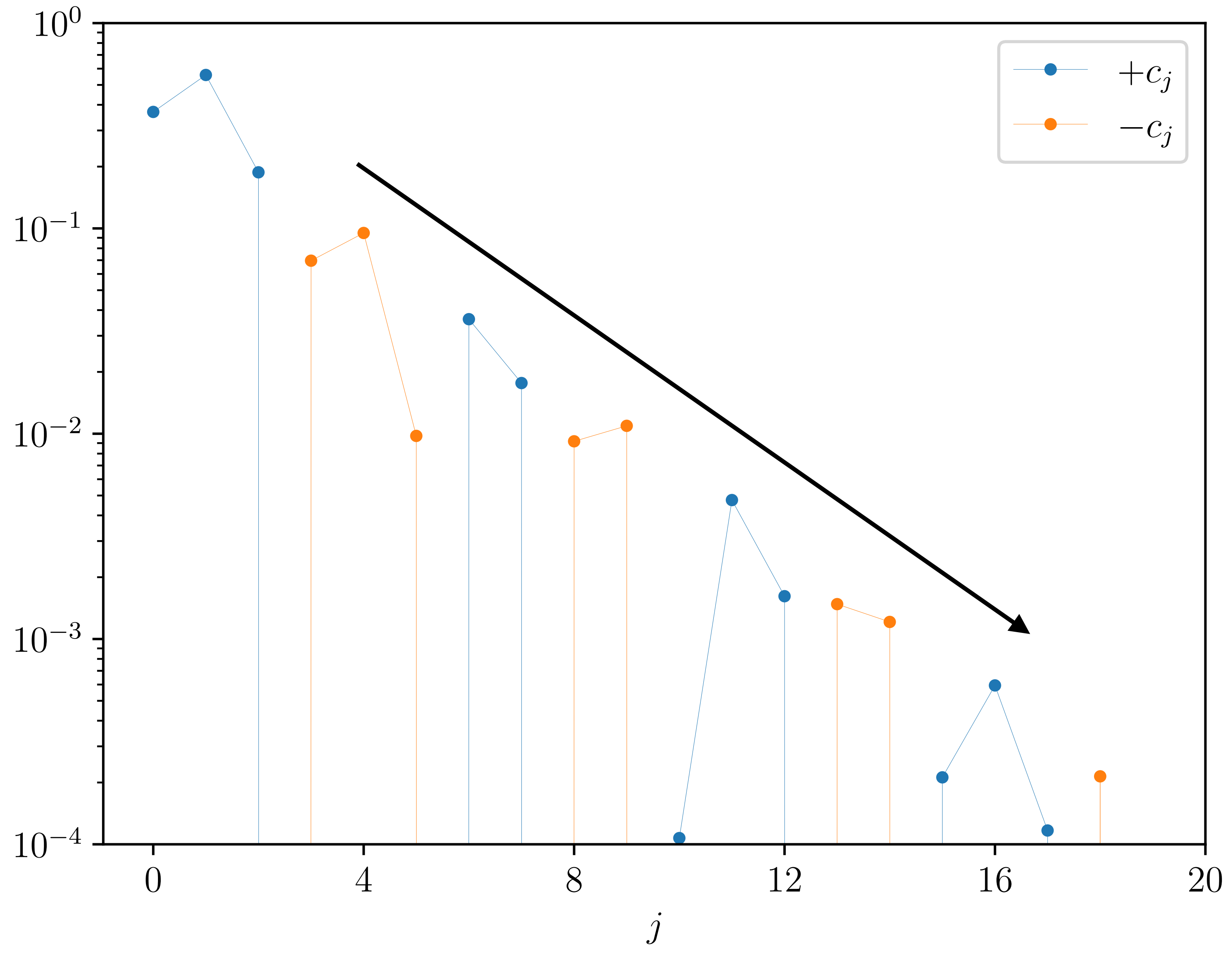
$\Delta = 0.01$



$\Delta = 0.05$



$\Delta = 0.1$



Exponentially converges. The smoother the kernel, the faster the convergence.

Possible to estimate the truncation error.

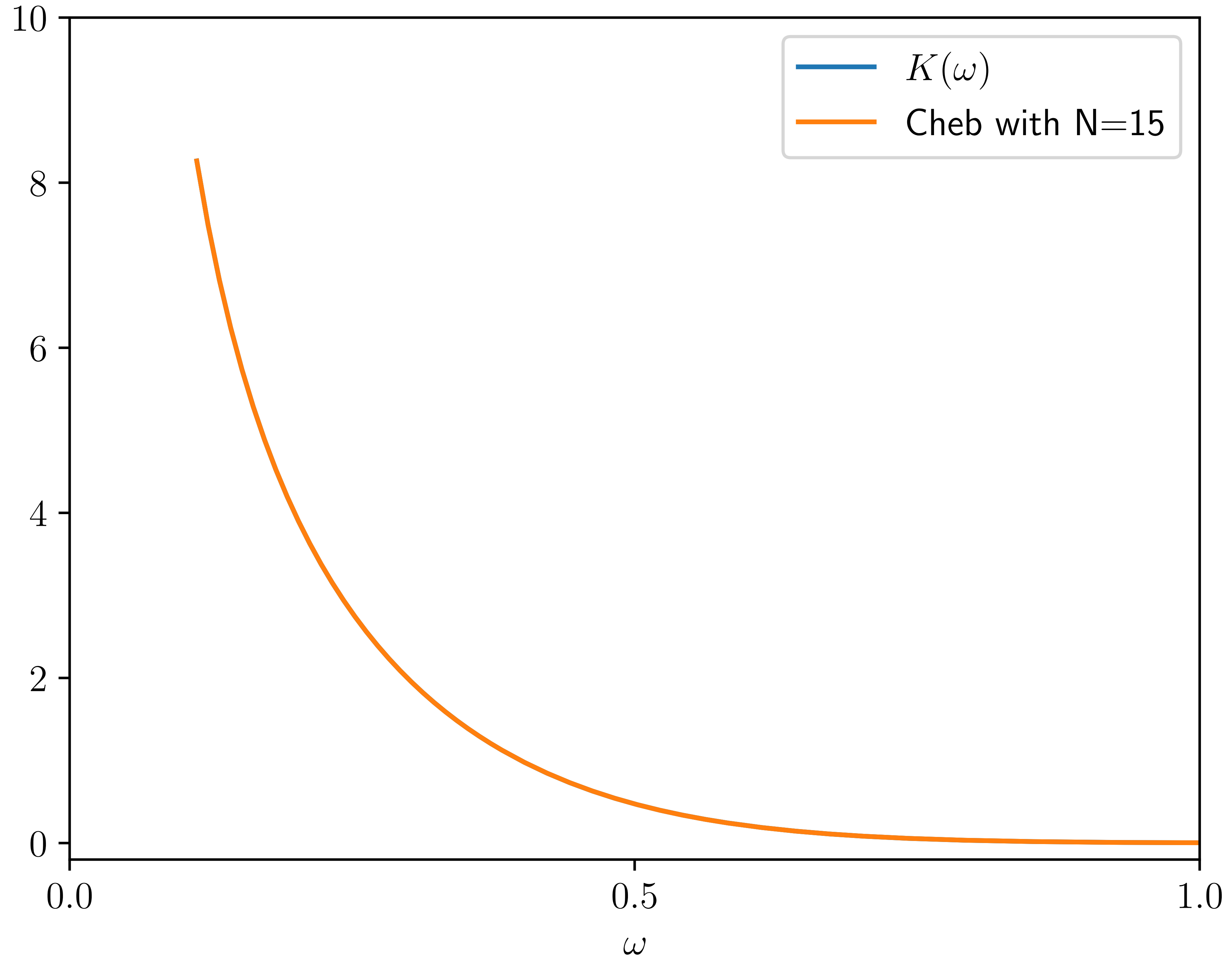
An example

Borel transform: Shifman, Vainshtein, Zakhalov (1979)

$$\tilde{\Pi}(M^2) = \frac{1}{M^2}\int_0^\infty ds\,\rho(s)e^{-s/M^2}$$
$$= \frac{1}{M^2}\int_0^\infty d\omega\,\frac{2}{\omega}e^{-\omega^2/M^2}\cdot\omega^2\rho(\omega^2)$$

from $C(t) = \int_0^\infty d\omega\,\omega^2\rho(\omega^2)e^{-\omega t}$

$$(q_\mu q_\nu - q^2 g_{\mu\nu})\Pi(q^2) = i\int d^4x\,e^{iqx}\langle J_\mu(x)J_\nu(0)\rangle$$
$$\rho(s) = \frac{1}{\pi}\mathrm{Im}\Pi(s+i\epsilon)$$



lattice unit,
with $M_0/\,a^{-1}$ = 1 GeV/2.4 GeV

Nearly perfect approx with *N*=15

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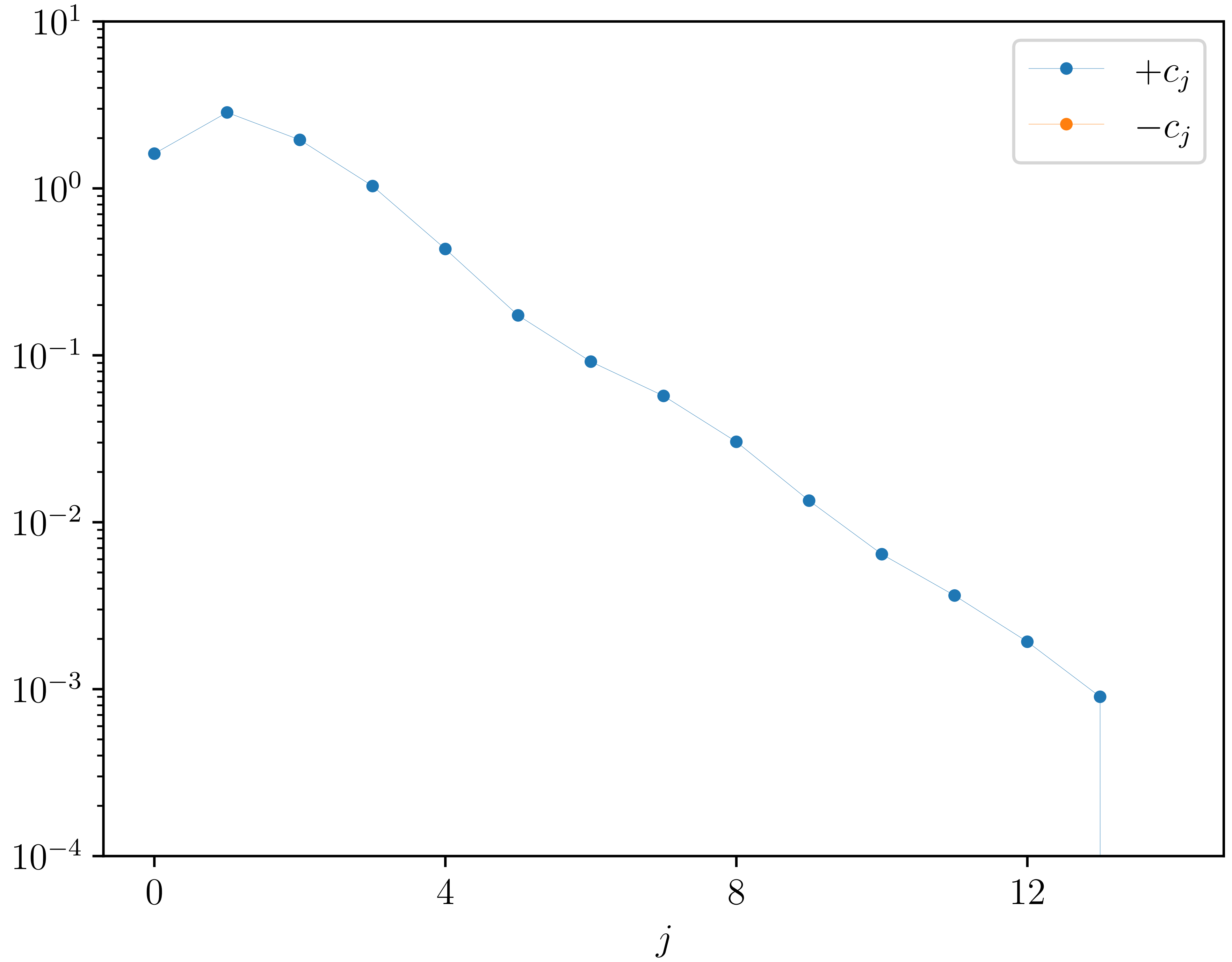


lattice unit,
with $M_0/\ a^{-1}$ = 1 GeV/2.4 GeV

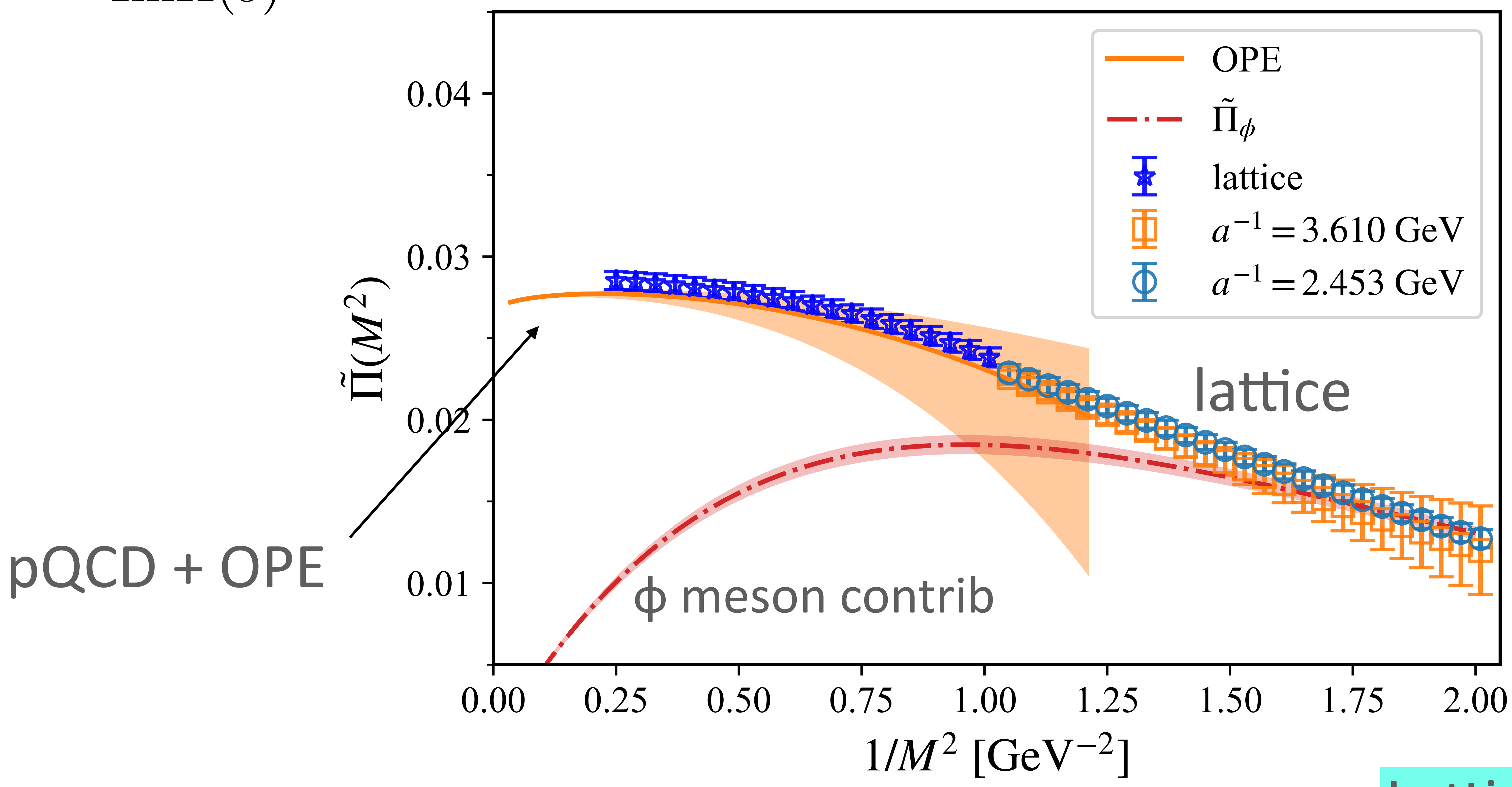
Nearly perfect approx with N=15

Borel transform (as in QCD sum rule)

Ishikawa, SH, Phys. Rev. D104, 074521 (2021)

$$\int ds\, e^{-s/M^2}\mathrm{Im}\Pi(s)$$

$s\bar{s}$ channel



Lattice can provide precise data in the entire energy range.

Chebyshev matrix elements

Shifted Chebyshev polynomials:

$$T_0^*(x) = 1$$
$$T_1^*(x) = 2x - 1$$
$$T_2^*(x) = 8x^2 - 8x + 1$$
$$T_3^*(x) = 32x^3 - 48x^2 + 18x - 1$$
$$\cdots$$

$$K(\omega) \simeq \frac{c_0}{2} + \sum_{j=1}^{N-1} c_j T_j^*(e^{-\omega})$$

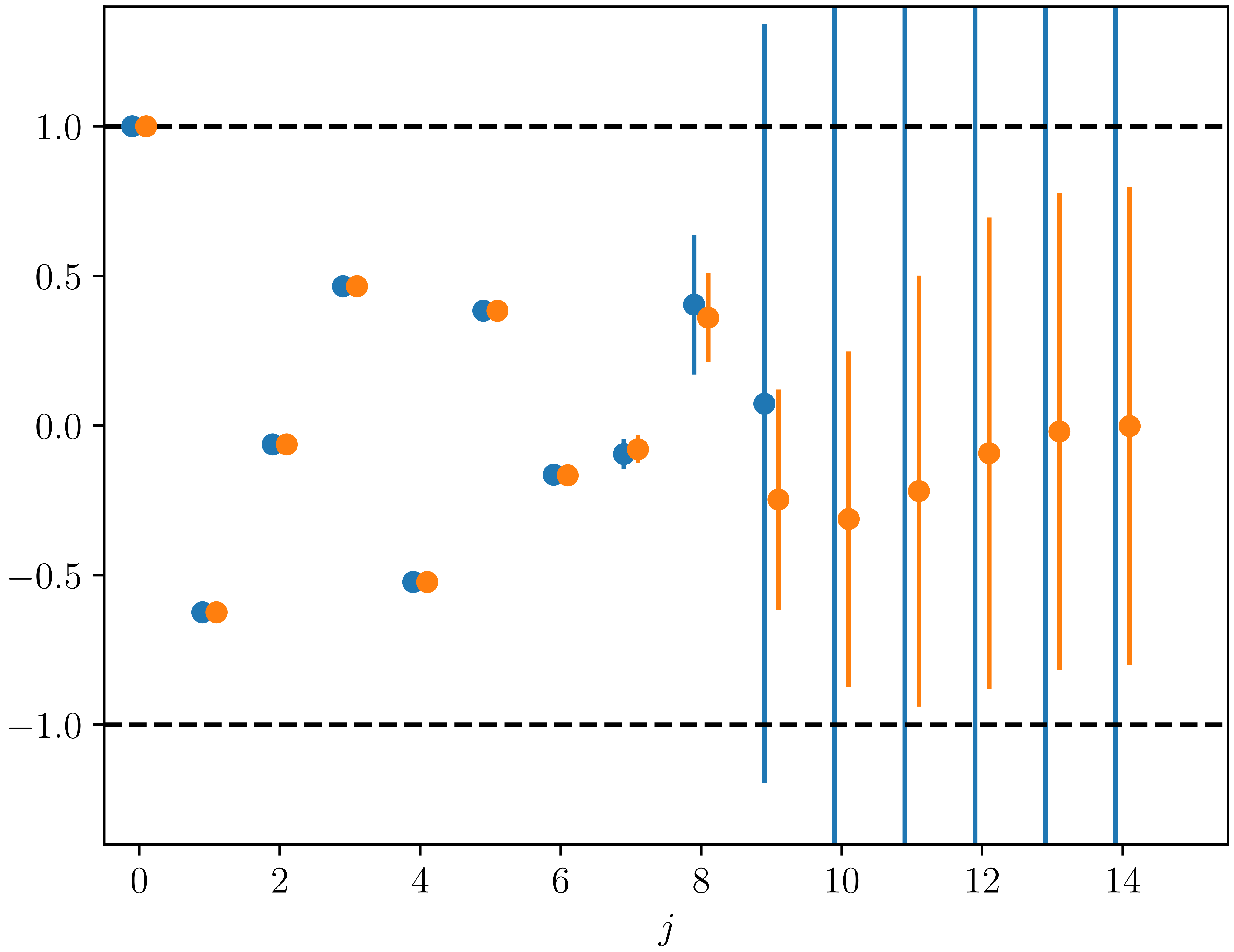
Matrix elements:

$$\bar{C}(t) \equiv \frac{C(t+t_0)}{C(t_0)}$$

$$\langle T_0^*(e^{-\hat{H}})\rangle = 1$$
$$\langle T_1^*(e^{-\hat{H}})\rangle = 2\bar{C}(1) - 1$$
$$\langle T_2^*(e^{-\hat{H}})\rangle = 8\bar{C}(2) - 8\bar{C}(1) + 1$$
$$\langle T_3^*(e^{-\hat{H}})\rangle = 32\bar{C}(3) - 48\bar{C}(2) + 18\bar{C}(1) - 1$$
$$\cdots$$

$$\langle K(\hat{H})\rangle \simeq \frac{c_0}{2} + \sum_{j=1}^{N-1} c_j \langle T_j^*(e^{-\hat{H}})\rangle$$

$\langle T_j^*(e^{-\hat{H}})\rangle$



Lattice data from JLQCD:

- Domain-wall, 48^3x96, a^{-1} ~ 2.45 GeV
- m_π ~ 230 MeV
- Vector channel, LL

Orange: fit with the reverse formula + a constraint $|T^*_j(x)| \leqq 1$

$$\bar{C}(n) = \sum_{j=0}^{n} d_j \langle T_j^*(e^{-\hat{H}})\rangle$$

Data soon become very noisy; the expansion possible only up to N = O(10).

Inclusive rate

Gambino, SH, Phys. Rev. Lett. 125 (2020) 032001
see also, Hansen, Meyer, Robaina, Phys. Rev. D96, 094513 (2017)

Semi-leptonic B/D decays:

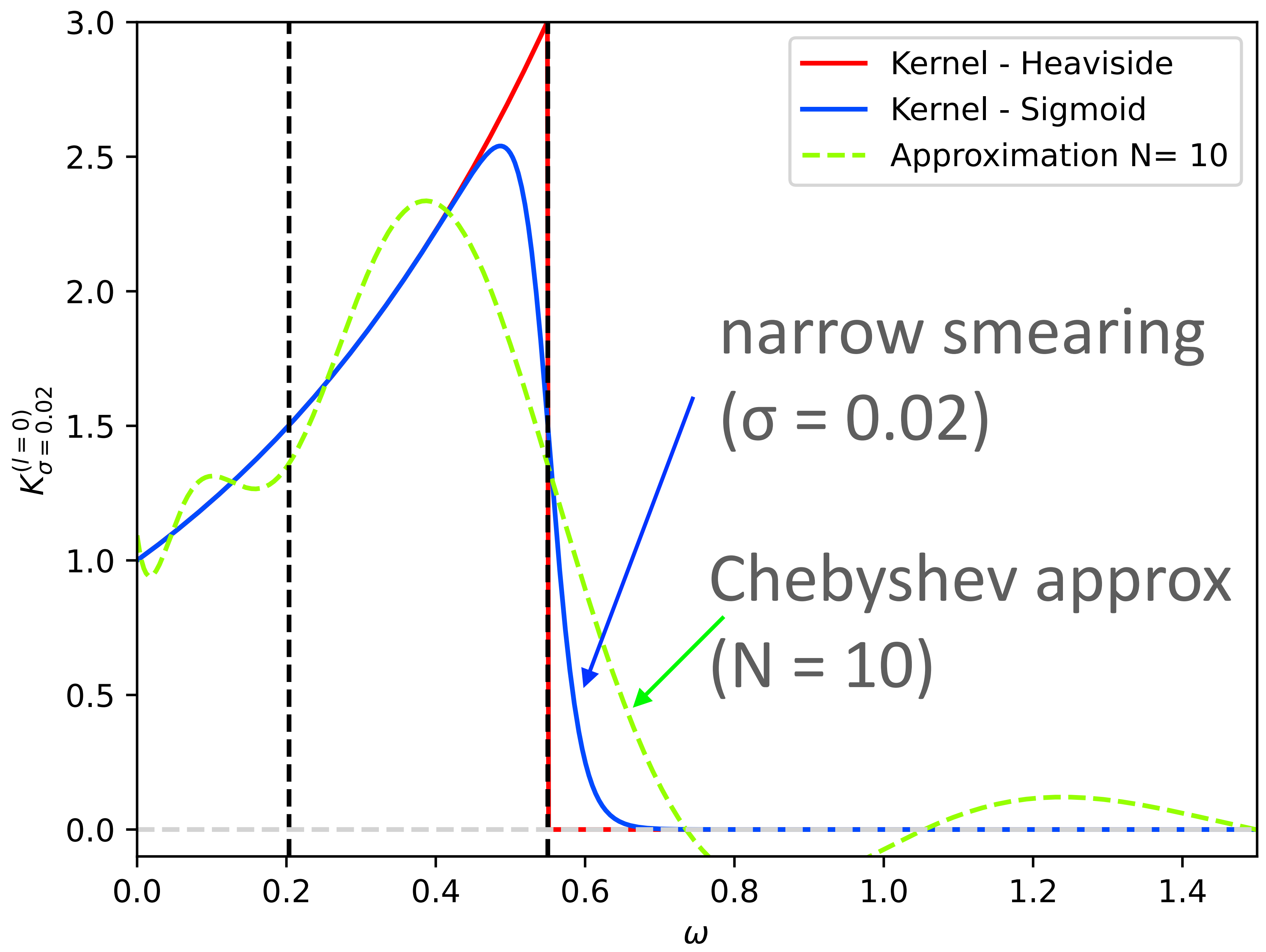
$$\Gamma \propto \int_0^{\boldsymbol{q}^2_{\max}} d\boldsymbol{q} \boxed{\int_{\sqrt{m_D^2+\boldsymbol{q}^2}}^{m_B-\sqrt{\boldsymbol{q}^2}} d\omega\, K(\omega;\boldsymbol{q}^2)\langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})\delta(\omega-\hat{H})\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle}$$

$$= \langle B(\mathbf{0})|\tilde{J}^\dagger(-\boldsymbol{q})K(\hat{H};\boldsymbol{q}^2)\tilde{J}(\boldsymbol{q})|B(\mathbf{0})\rangle$$

Kernel:

$$K(\omega) \sim e^{2\omega t_0}(m_B-\omega)^l\theta(m_B-|\mathbf{q}|-\omega)$$

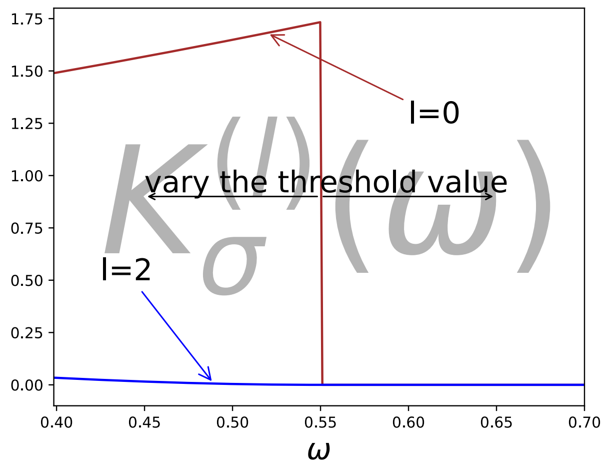
- Need to smear the step function to obtain reasonable approximation, with σ = 1/N, say.
- Extrapolation to N→∞, see below.

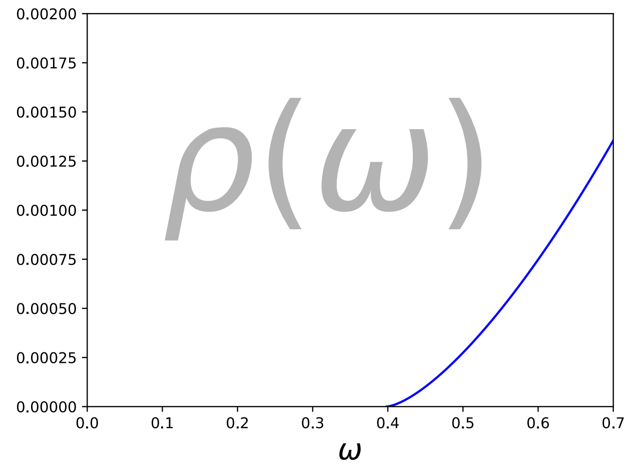


Systematic errors?

- Differential decay rate:

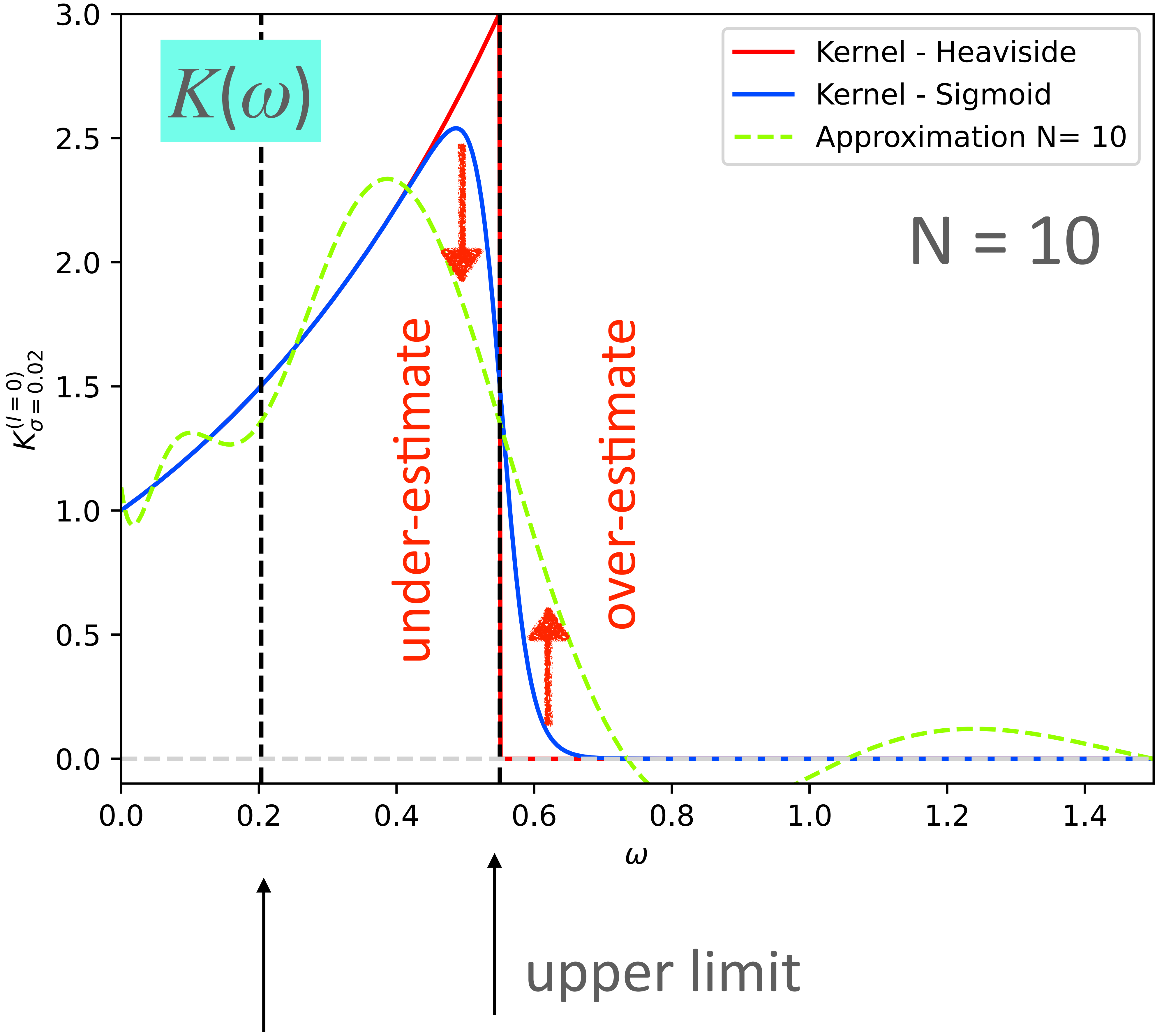
$$\bar{X}^{(l)}(\omega_{th}) \sim \int_0^{\omega_{th}} d\omega$$





  - Have to truncate the expansion.
  - We don’t know the spectrum a priori.

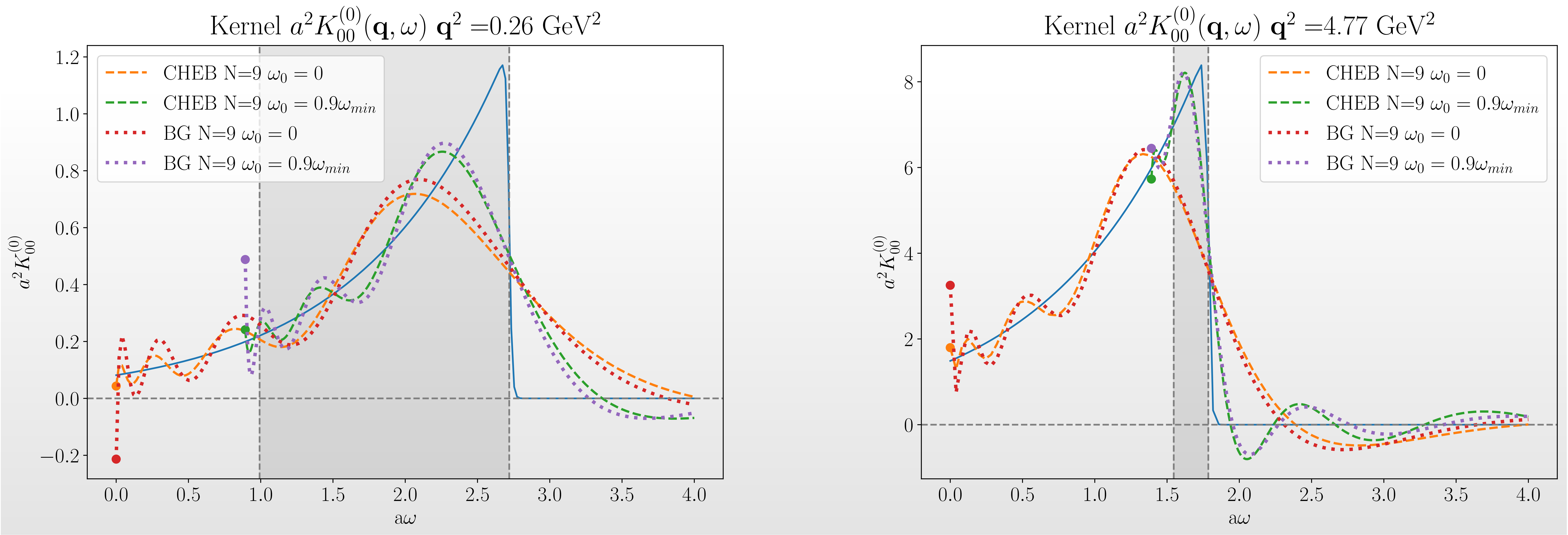
narrow smearing (σ = 0.02)



upper limit

lowest energy state

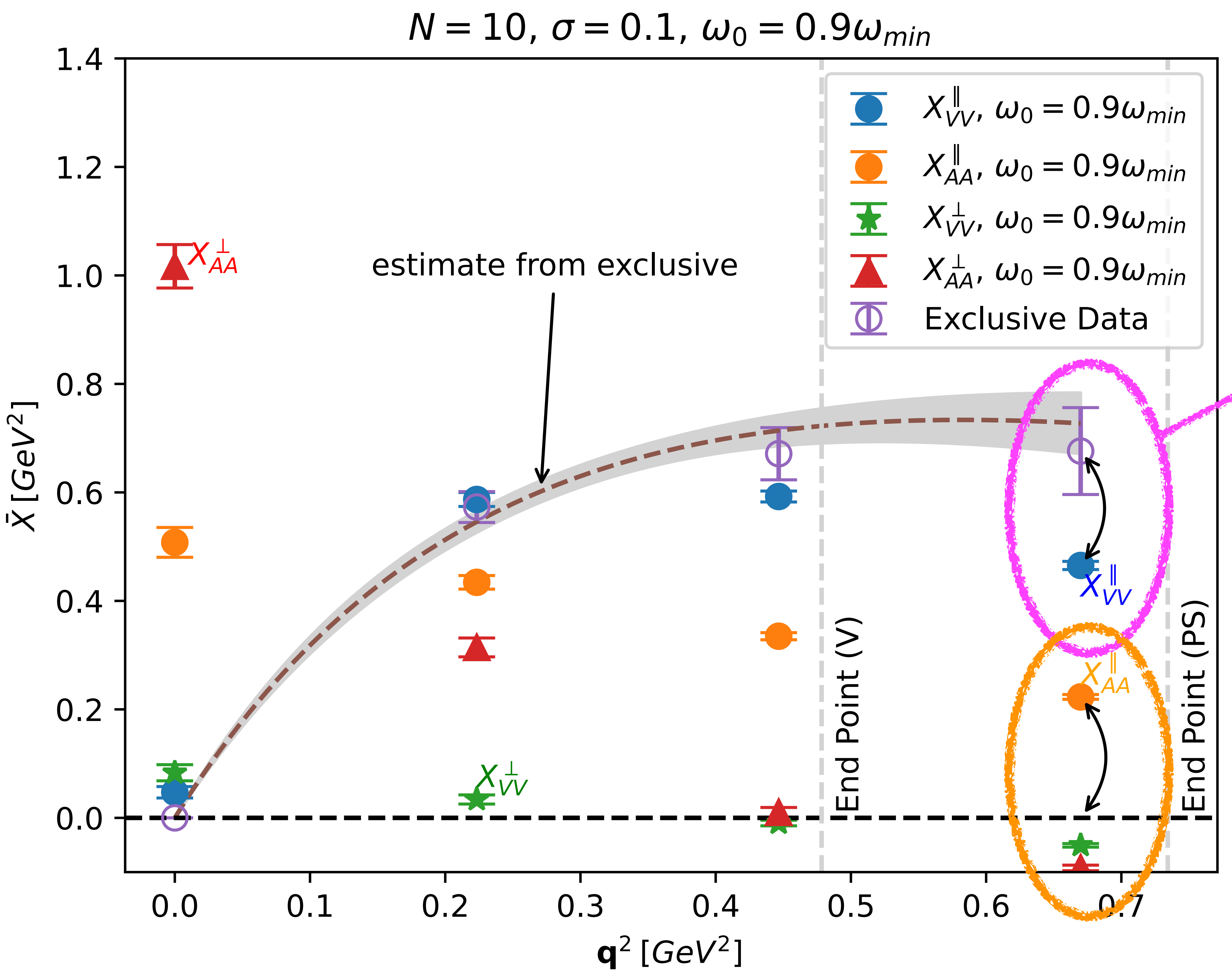
Barone et al. , JHEP 07 (2023) 145; arXiv:2305.14092



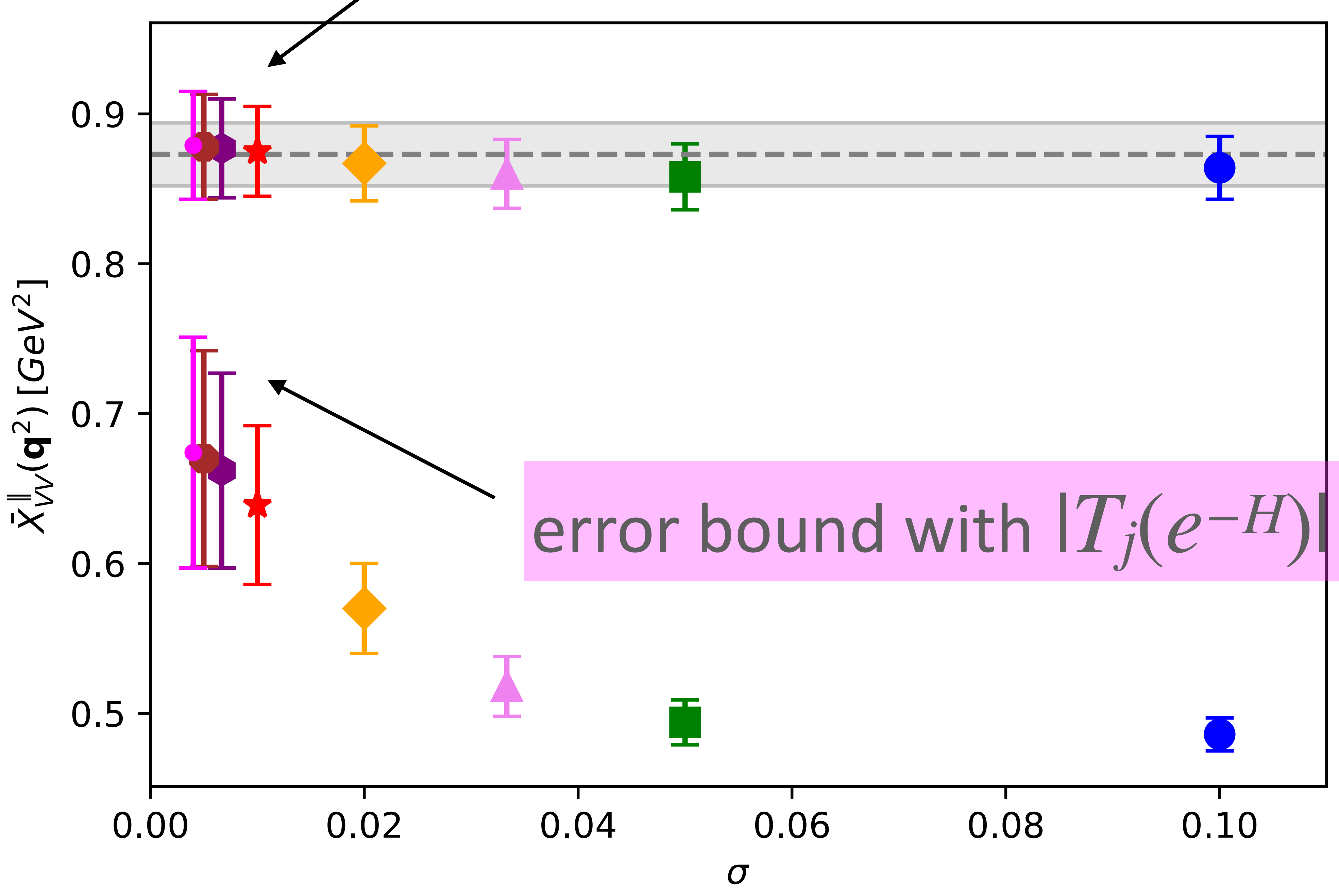
- Relevant energy range depends on the momentum transfer.
- Can improve by adjusting the lower limit of the approximation.
- Backus-Gilbert and Chebyshev give essentially the same approx.

Truncation error: the worst case

Ds semi-leptonic decays:
Kellermann @ Lattice 2022



ground state separately treated

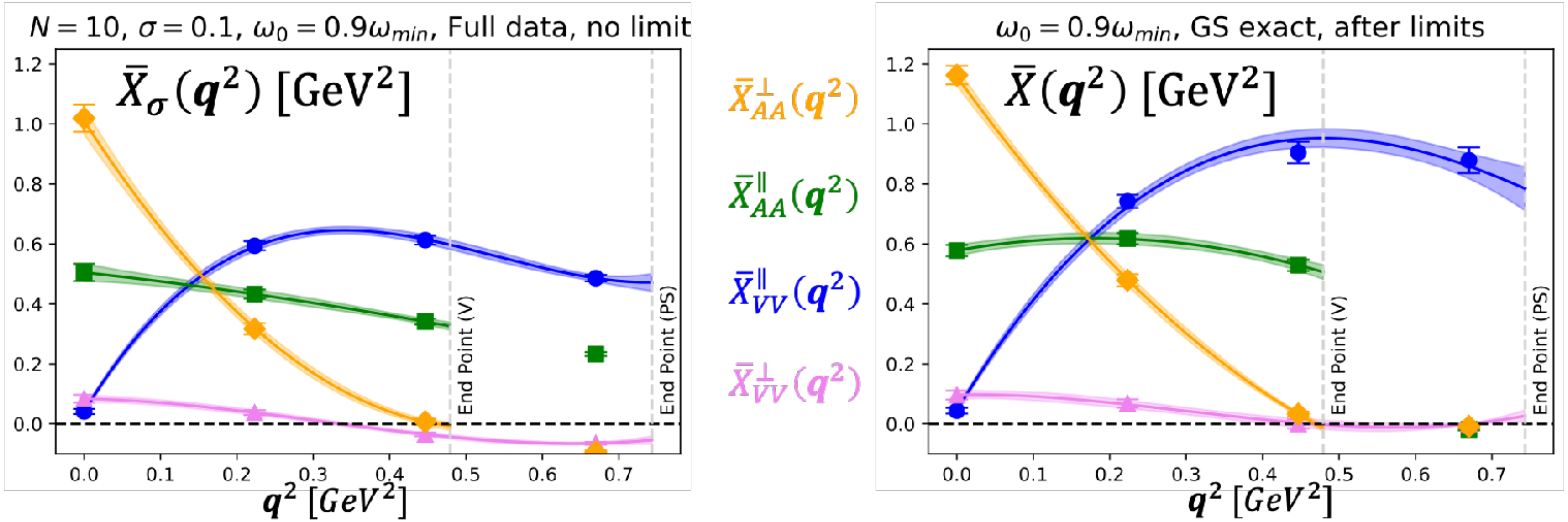


error bound with $|T_j(e^{-H})| \leqq 1$

increasing order of poly with σ = 1/N

Dangerous near the kinematical end-point.
Ground-state can be treated exactly, anyway.

D_s semi-leptonic decays:
Kellermann @ Lattice 2024



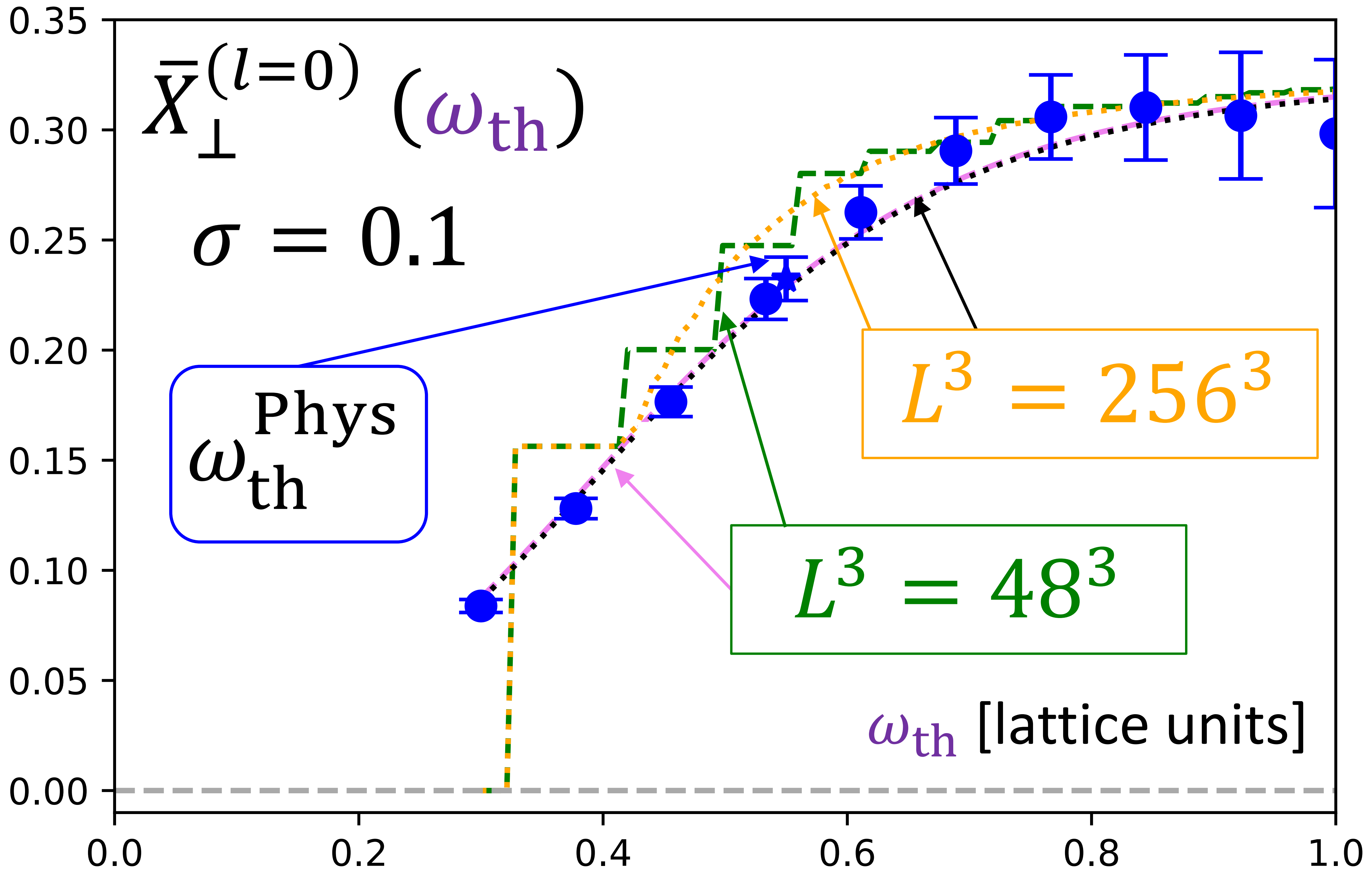
Fixed smearing width; fixed polynomial order
$\sigma = 1/N = 0.1$

Ground-state treated exactly. Chebyshev approx the rest. Added to all orders; error is estimated using $|T^*_j(x)| \leqq 1$.

Finite volume effect?

- Two-body state contribution may induce power-law, $1/L^{\alpha}$, corrections.
- Can be estimated using models (form factors). Not significant in this particular case.

$X_{AA}^{\perp}$ ($\mathbf{q}^2$) for $\mathbf{q}$=(0,0,0)



A model with

$$\langle K(\boldsymbol{p})\bar{K}(\boldsymbol{p}')|\tilde{J}^{\mu}(\boldsymbol{q})|D_s\rangle \sim (\boldsymbol{p}-\boldsymbol{p}')^{\mu}F(\omega,\boldsymbol{q})$$

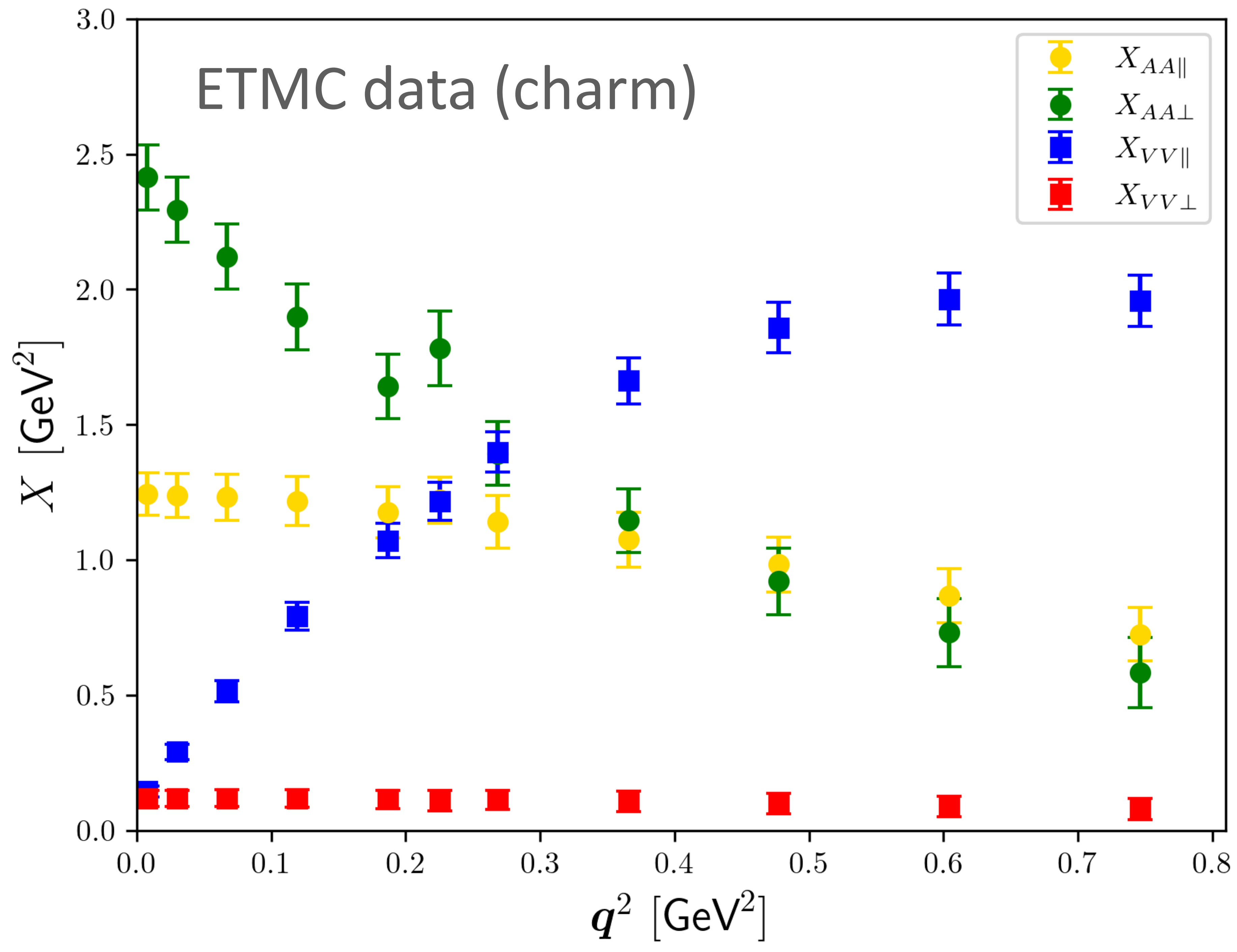
studies with varying upper limit ω_{th}

Its physical value is given by the released energy.

Results so far:

Full error analysis still to be done

Gambino et al. (2022),
JHEP 07 (2022) 083; arXiv:2203.11762



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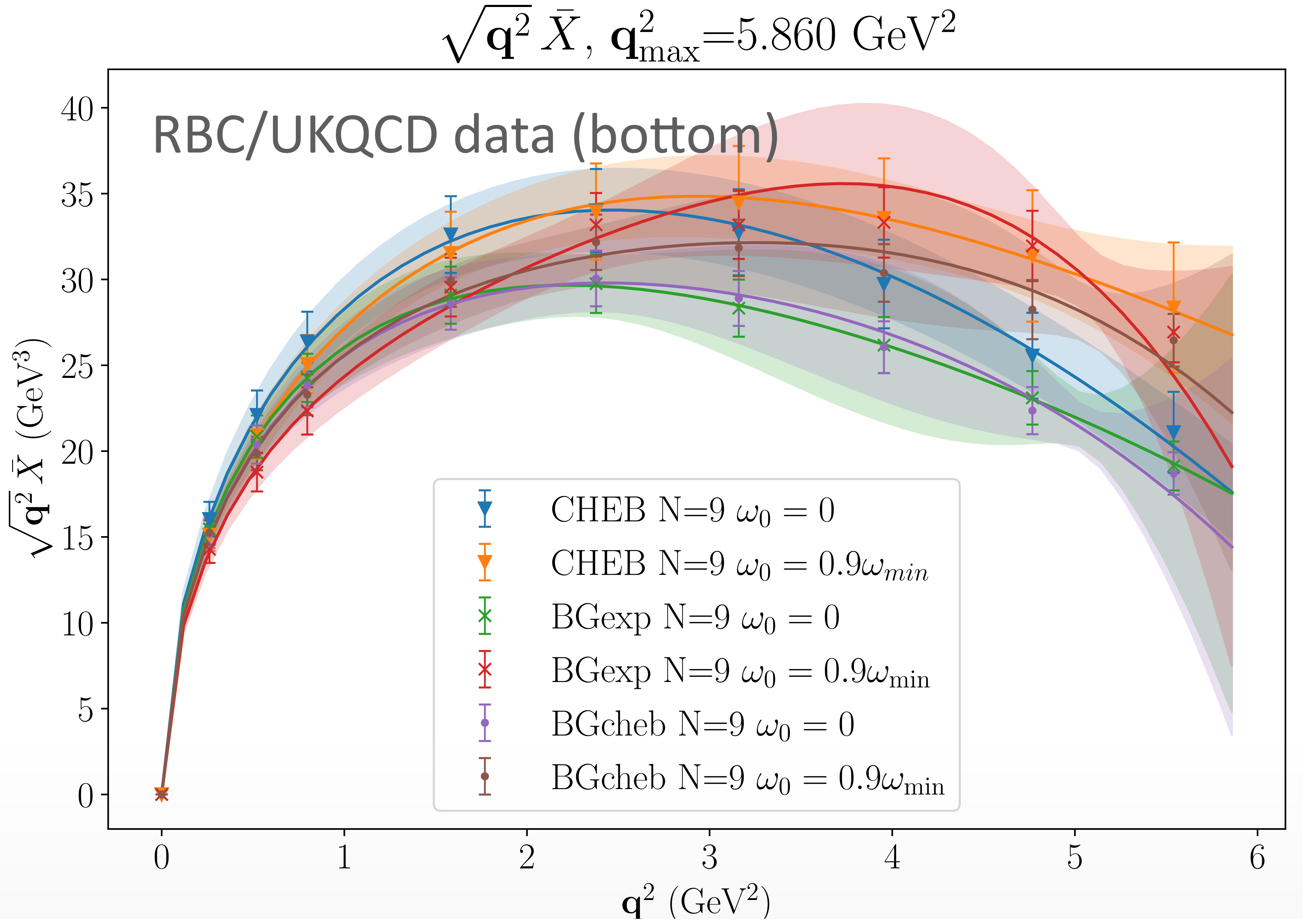

Summary

- Inclusive processes as a smeared spectrum: lattice calculation is possible with the kernel approximation.
- Systematic errors
  - Truncation errors: upper-limit can be estimated (Chebyshev)
  - Finite volume: need some estimates
- Complementary to Luschers's finite-volume approach
  - No need to identify each energy level: no info of scattering phase shifts obtained.