Tensor network study of CP(1) model with theta-term

Shinji Takeda

(Kanazawa U.)

In collaboration with H. Aizawa and Y. Yoshimura

German Japanese Workshop 2024

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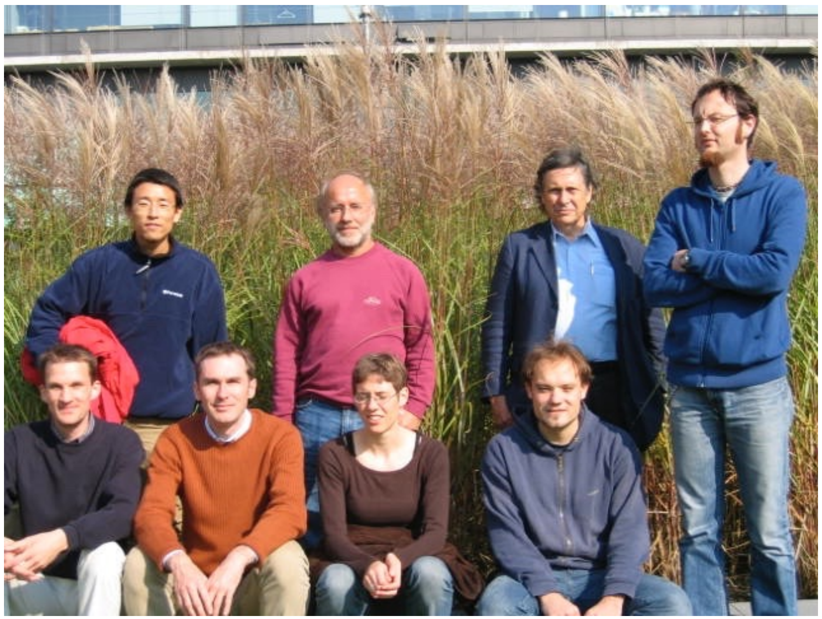
2005-2008 Postdoc time



Contents

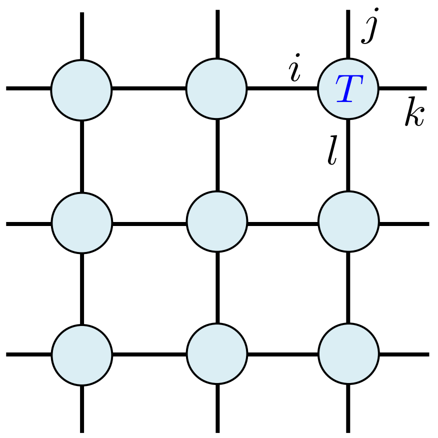
- Introduction of tensor networks
- Application to CP(1) + θ model

Tensor networks (TN)

- Why TN? Answer: No sign-problem

  Real-time, Finite density, θ-term, …

  we address later
- What's TN?



$$= \sum_{..,i,j,k,l,..} \cdots T_{ijkl} T_{mnio} \cdots$$

- TN is used as a representation of wave function or partition function (path integral)

How to compute Z using TN

local interaction

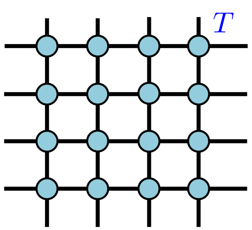
$$Z \equiv \int [d\phi] e^{-S[\phi]}$$

How to compute Z using TN

local interaction

$$Z \equiv \int [d\phi] e^{-S[\phi]} = \sum_{..,i,j,k,l,..} \cdots T_{ijkl} T_{mnio} \cdots$$

Periodic BC

$=$ 

- Non-trivial step, but
- OK for scalar, gauge, and fermion fields as long as the interaction is local

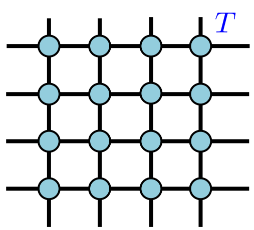
- Scalar : Shimizu 2012, Sakai+ 2018
- Gauge: Meurice+ 2013, Nishimura+ 2021, Fukuma+ 2021
- Fermions: Shimizu+Kuramashi 2014

How to compute Z using TN

$$Z \equiv \int [d\phi] e^{-S[\phi]} = \sum_{\dots,i,j,k,l,\dots} \cdots T_{ijkl} T_{mnio} \cdots$$

Periodic BC

=



cost $\propto \chi^{2V}$ for $1 \leq i, j, \dots \leq \chi$

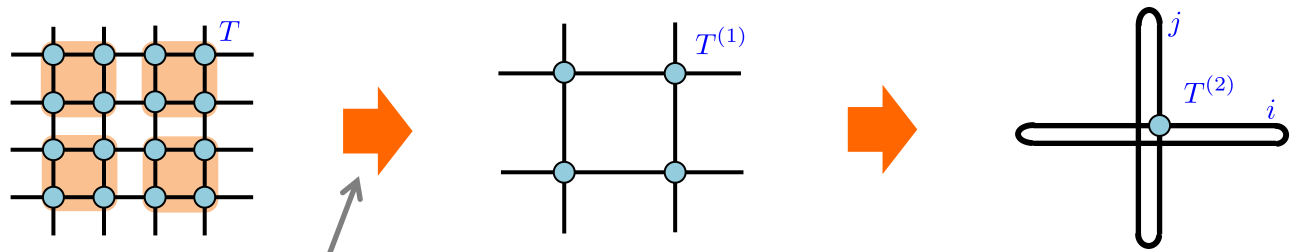
Bond dimension

How to compute Z using TN

$$Z \equiv \int [d\phi] e^{-S[\phi]} = \sum_{..,i,j,k,l,..} \cdots T_{ijkl} T_{mnio} \cdots$$

Periodic BC

=



Coarse-graining like spin-blocking

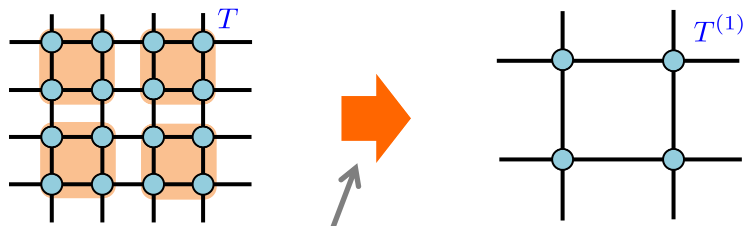
$$Z \approx \sum_{i,j} T^{(n)}_{ijij}$$

Information compression using
singular value decomposition (SVD)

Coarse-graining

Tensor renormalization group (TRG)
Levin+Nave 2007

Z =

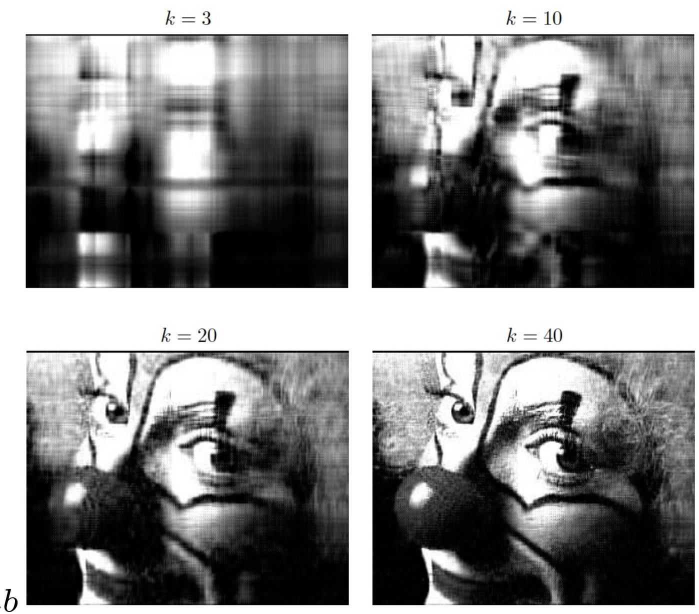


Information compression using
singular value decomposition (SVD)

original

truncated SVD

less modes



more modes

SVD: $M_{ab} = \sum_m u_{am} \Lambda_m (v^\dagger)_{mb} \approx \sum_{m=1}^{k} u_{am} \Lambda_m (v^\dagger)_{mb}$

$\Lambda_0 \geq \Lambda_1 \geq \dots$

http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT

Application to $CP(1) + \theta$

2d CP(1) model

toy model of QCD

- asymptotic free
- confinement
- instanton

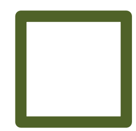
continuum

$$S = \int d^2x \left(\beta |D_\mu z|^2 + i\frac{\theta}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu \right)$$

U(1) auxiliary field

$$z \in \mathbb{C}^2$$

$$|z|^2 = \sum_{a=1}^{2} z_a^* z_a = 1$$



mod 2π

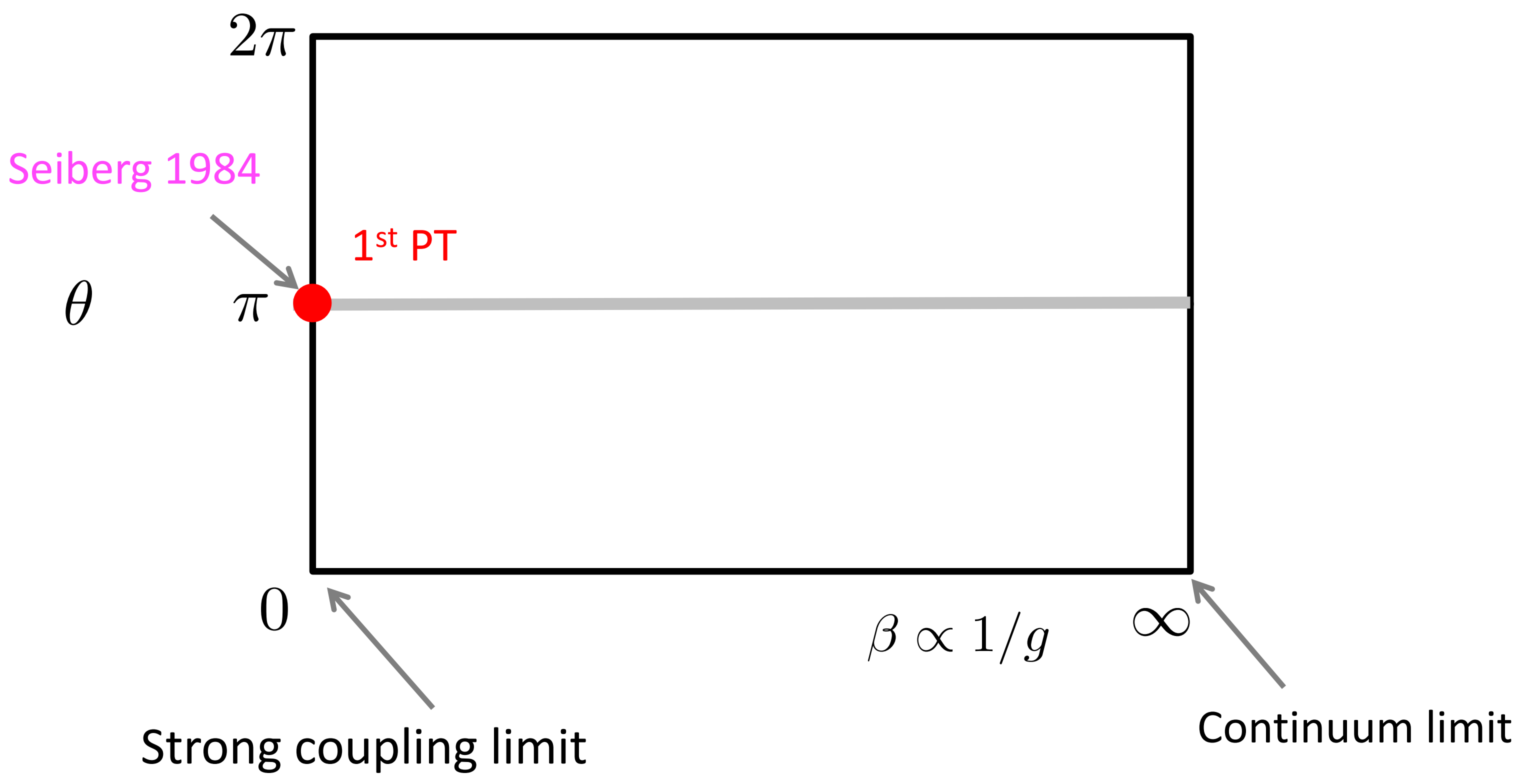
on the lattice

$$S_{\mathrm{lat}} = -2\beta \sum_{x,\mu} \left[z^\dagger(x) z(x+\hat{\mu}) U_\mu(x) + (c.c.) \right] - i\frac{\theta}{2\pi}\sum_x q(x) \quad \in \mathbb{C}$$

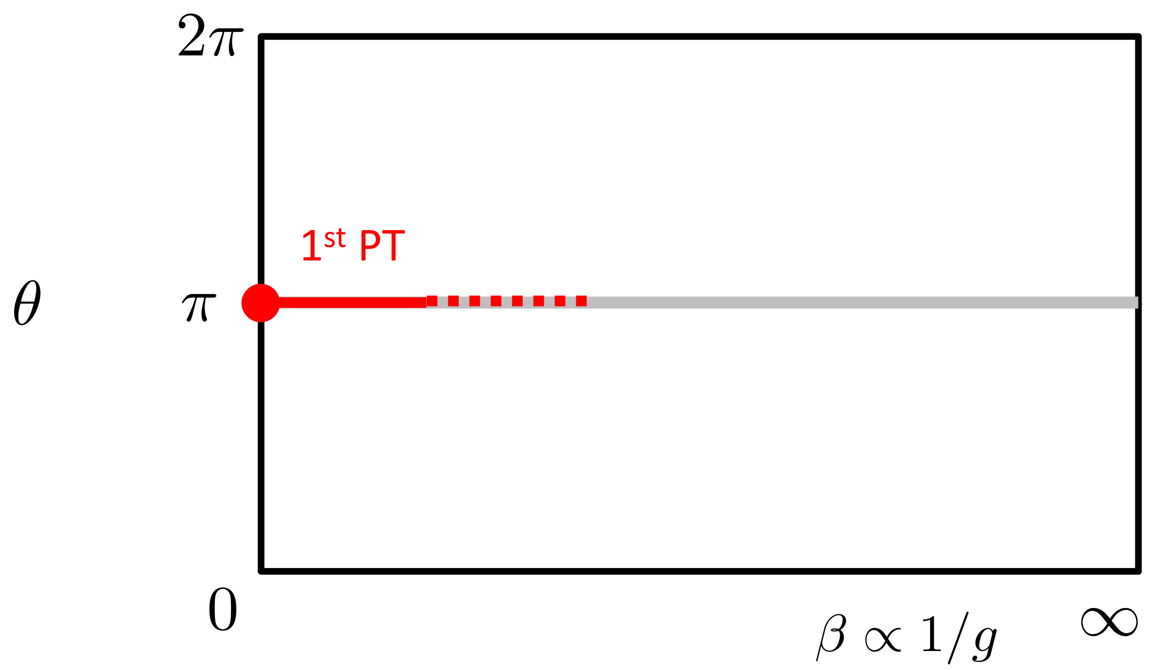
$$U_\mu = e^{iA_\mu}$$

sign problem!

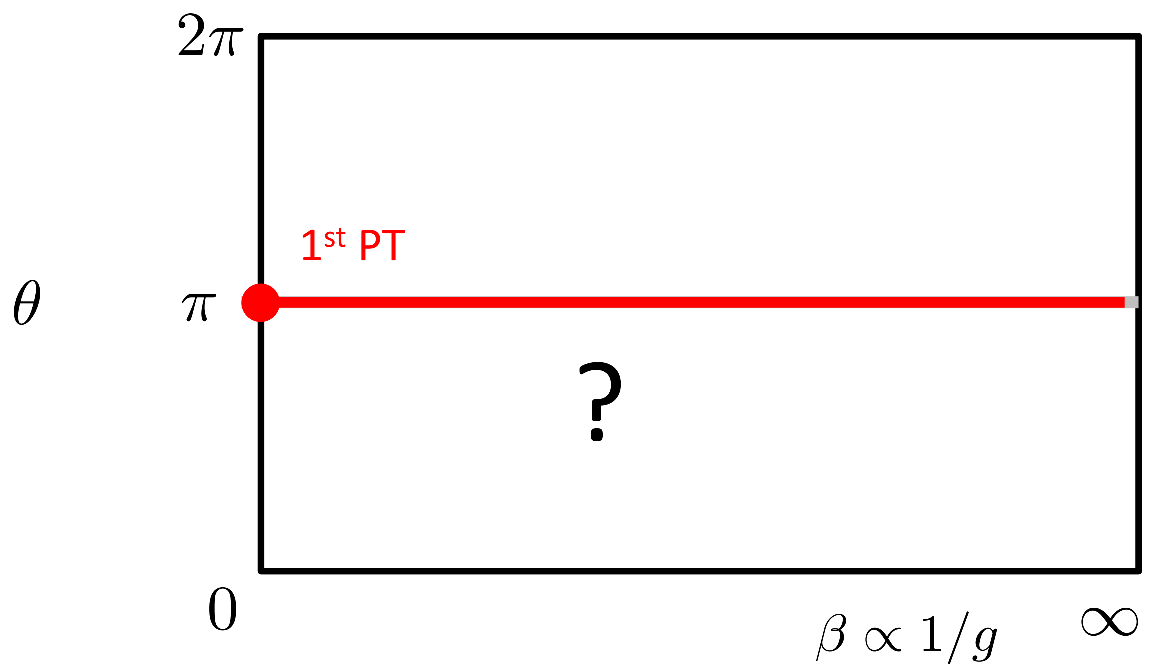
Previous study of phase structure



Previous study of phase structure



Previous study of phase structure



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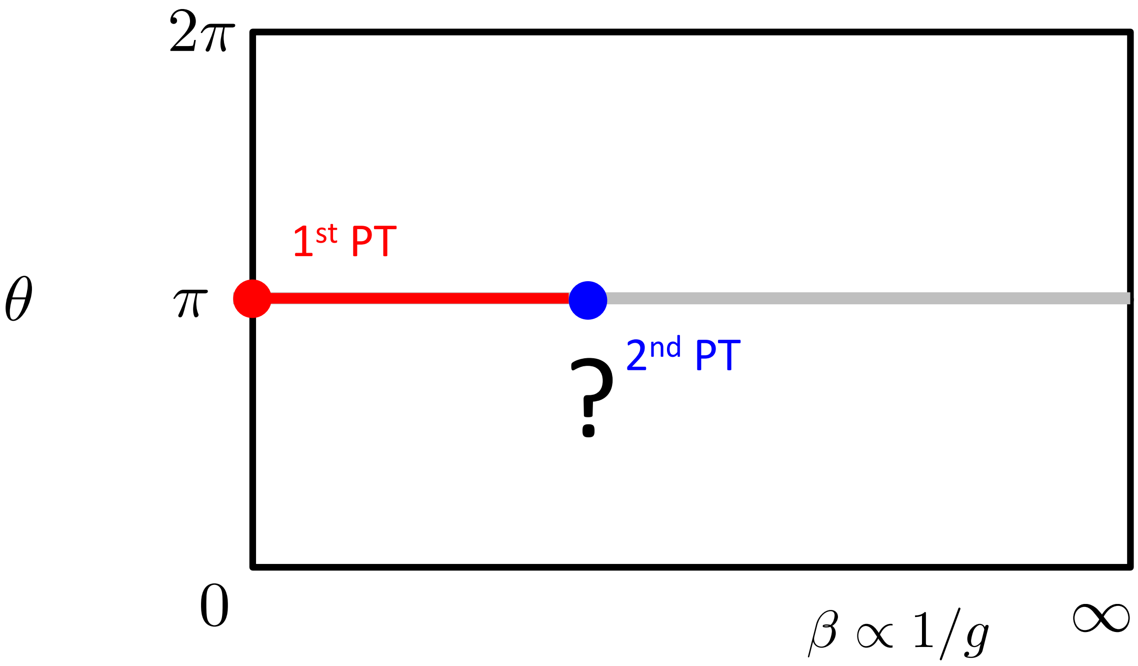
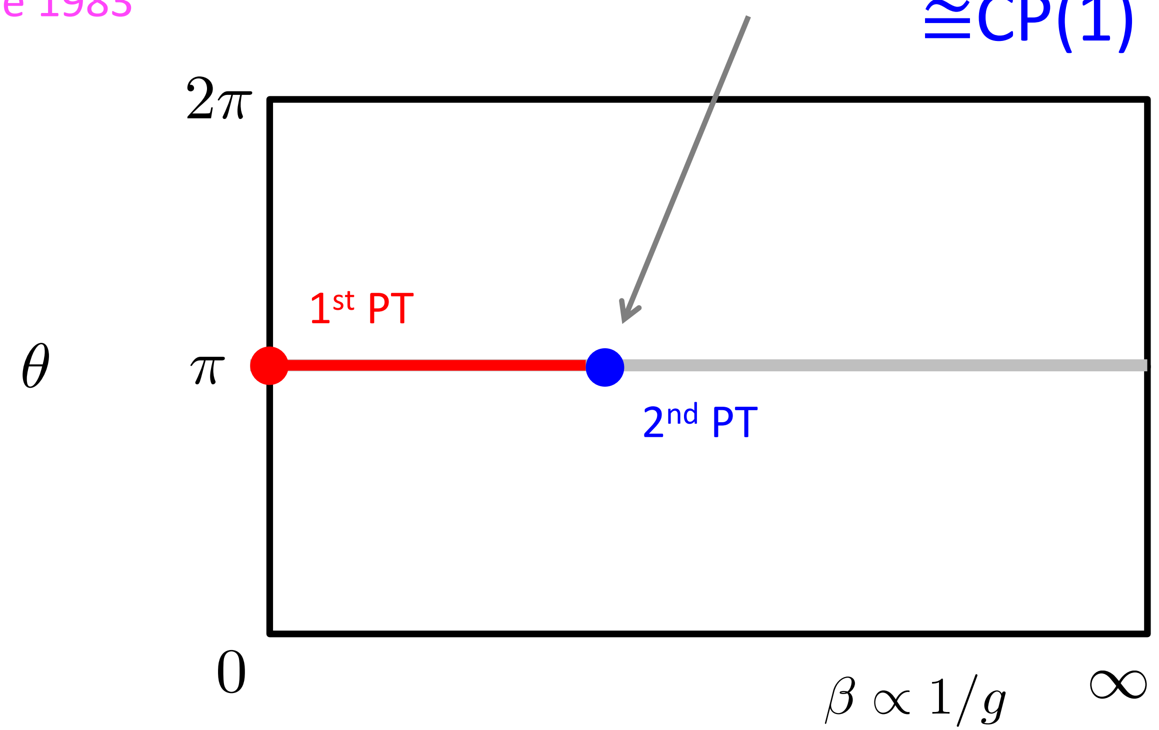


Previous study of phase structure

Haldane's conjecture : mass gap of O(3) vanishes at θ=π

Haldane 1983

≅CP(1)

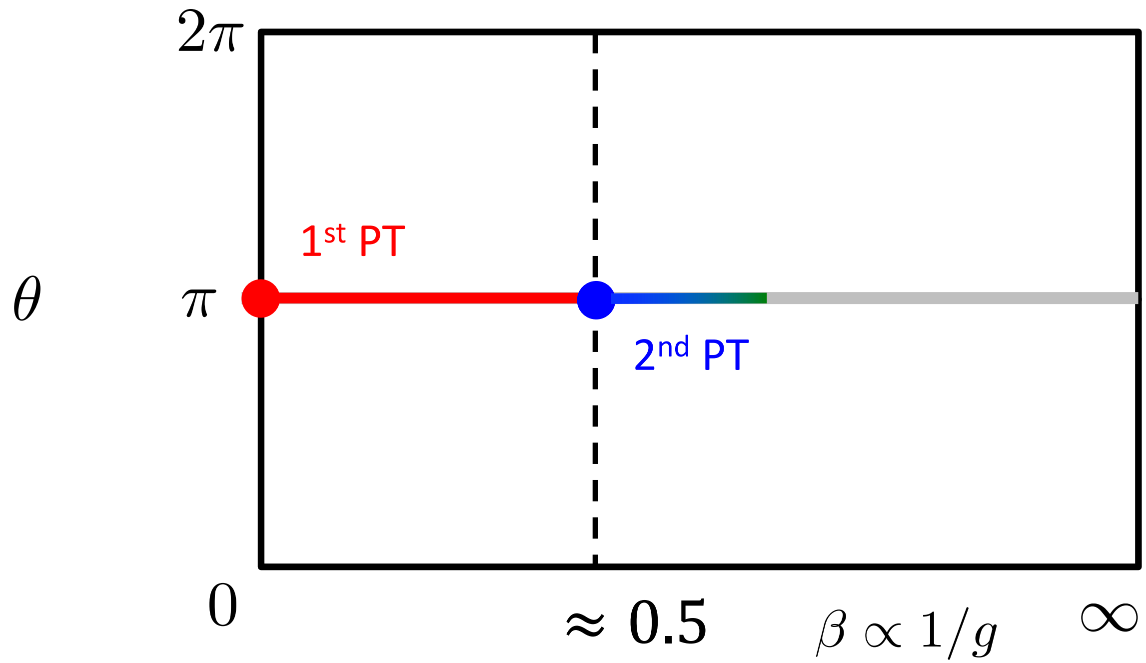


Affleck+Haldane 1987 expected universality class : $k=1$ SU(2) WZNW

Wess+Zumino 1971, Novikov 1981, Witten 1984

Previous study of phase structure

Azcoiti+ 2007 Monte Carlo, imaginary θ



Critical region is observed but the universality class is not identified

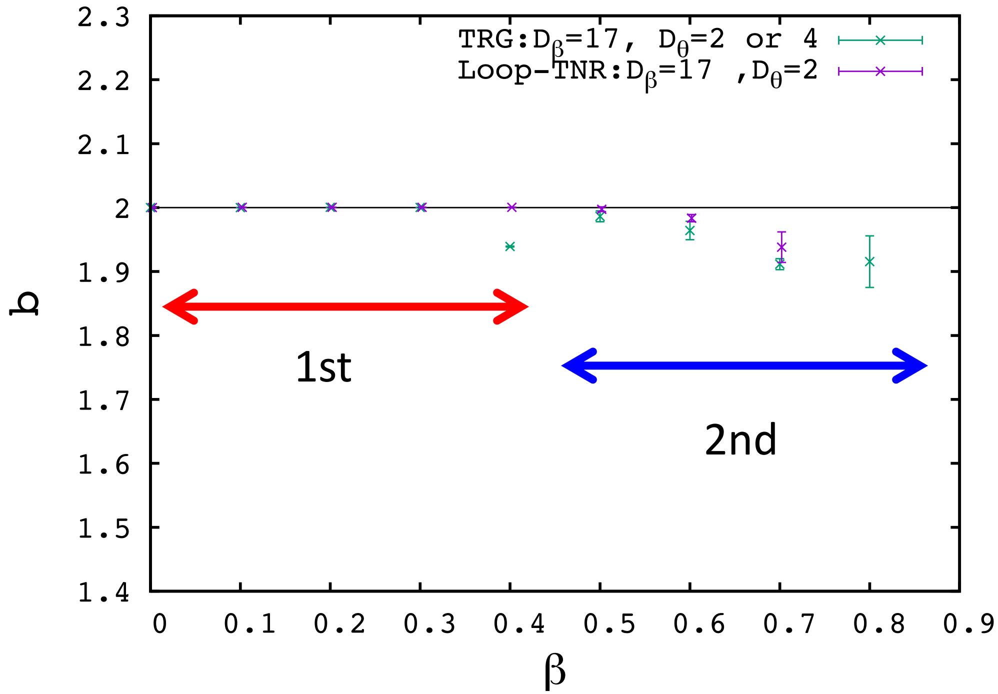
⇒ This should be investigated by sign problem free method : tensor network!

Previous study of phase structure

Kawauchi+ST 2018

Tensor network

$$\chi = \frac{1}{L^2}\frac{\partial^2}{\partial\theta^2}\log\mathcal{Z}(\theta)|_{\theta=\pi} \propto L^b$$



$$1\text{st} \quad : b = 2$$
$$2\text{nd} \quad : b = \gamma/\nu < 2$$

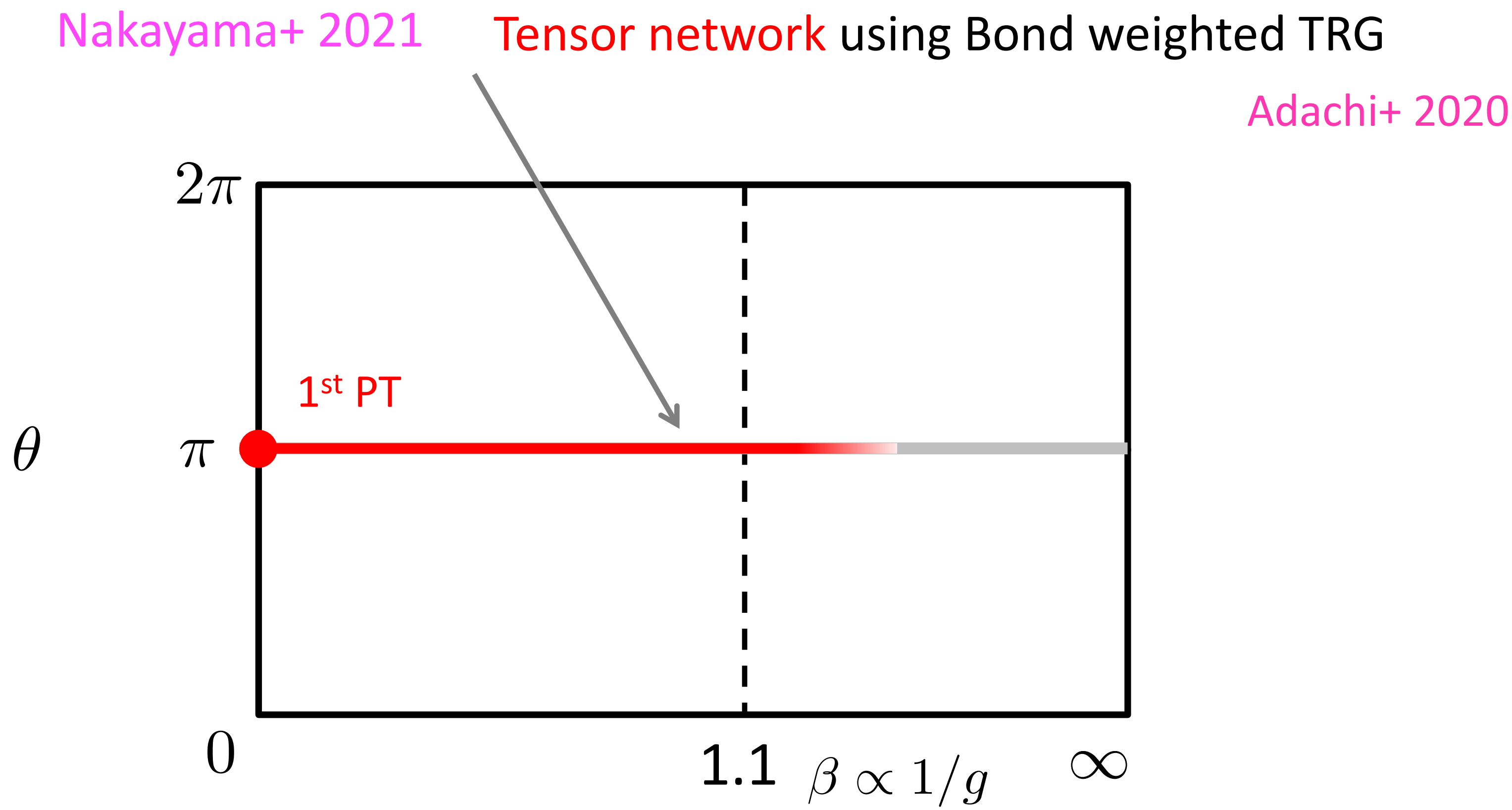
$\beta = 0.5$ is roughly consistent with MC

Azcoiti+ 2007

But universality class is not determined

D_β , D_θ : truncation order

Previous study of phase structure



no critical point is found up to β=1.1

What’s new in our study

We make two improvements

- Using better initial tensor

- New analysis based on CFT

Improvement of initial tensor

Kawauchi+ST 2017, Nakayama+ 2021

- Previous studies use character expansion for θ-term

Hassan+ 1995

$$e^{i\frac{\theta}{2\pi}q_p} = \sum_{n\in\mathbb{Z}} e^{in(A_1+A_2-A_3-A_4)} C_n(\theta)$$

$C_n(\theta) \propto \frac{1}{n}$: slow convergence!

⇒ large truncation error

- New method : Quadrature

Gauss-Legendre quadrature

Gauge field : $\int_{-\pi}^{\pi} dA f(A) \approx \sum_{i=1}^{N_A} w_i f(A_i)$ $\quad$ i ⇒ tensor's index

w_i: weight

Complex scalar field : $\int_{|z|^2=1} dz g(z) \approx \sum_{p=1}^{N_z} w_p^{(z)} g(z_p)$ $\quad$ p ⇒ tensor's index

Genz+Keister 1996

Improvement of initial tensor

$$\delta f = \left| \frac{f_{\mathrm{tensor}} - f_{\mathrm{exact}}}{f_{\mathrm{exact}}} \right|$$

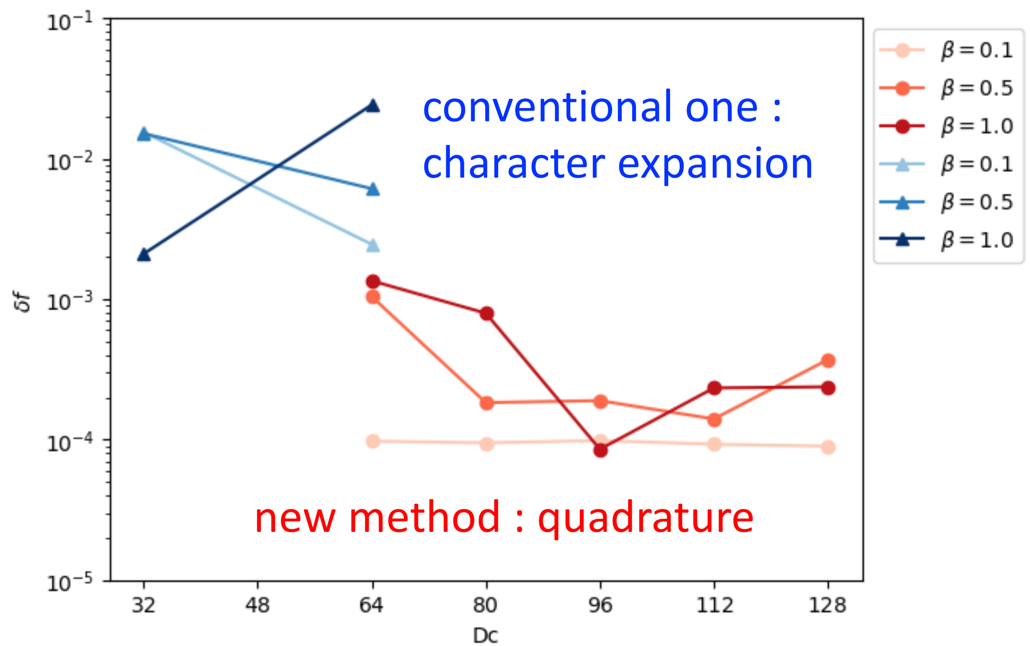
free energy

$V = 2 \times 2$

$\theta = \pi$

$N_A = 120$

$N_z = 224$



Bond dimension for initial tensor

New method shows better precision

New analysis method

- Previous studies use susceptibility

Kawauchi+ST 2017, Nakayama+ 2021

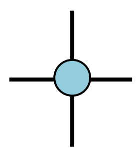
$$\chi = \frac{1}{L^2}\frac{\partial^2}{\partial\theta^2}\log\mathcal{Z}(\theta)|_{\theta=\pi} \propto L^b$$

fitting around $\theta = \pi$
$\Rightarrow$ fitting range ?

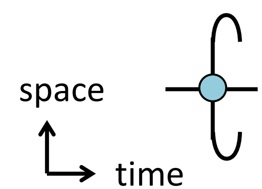
- CFT : central charge and scaling dimensions

Gu+Wen 2009

scale invariant tensor



$\Rightarrow$ scale invariant transfer matrix



PBC

EV of TM

$\Rightarrow \quad \lambda_0 \geq \lambda_1 \geq \dots \Rightarrow$

central charge

$$c = \frac{6}{\pi}\log(\lambda_0)$$

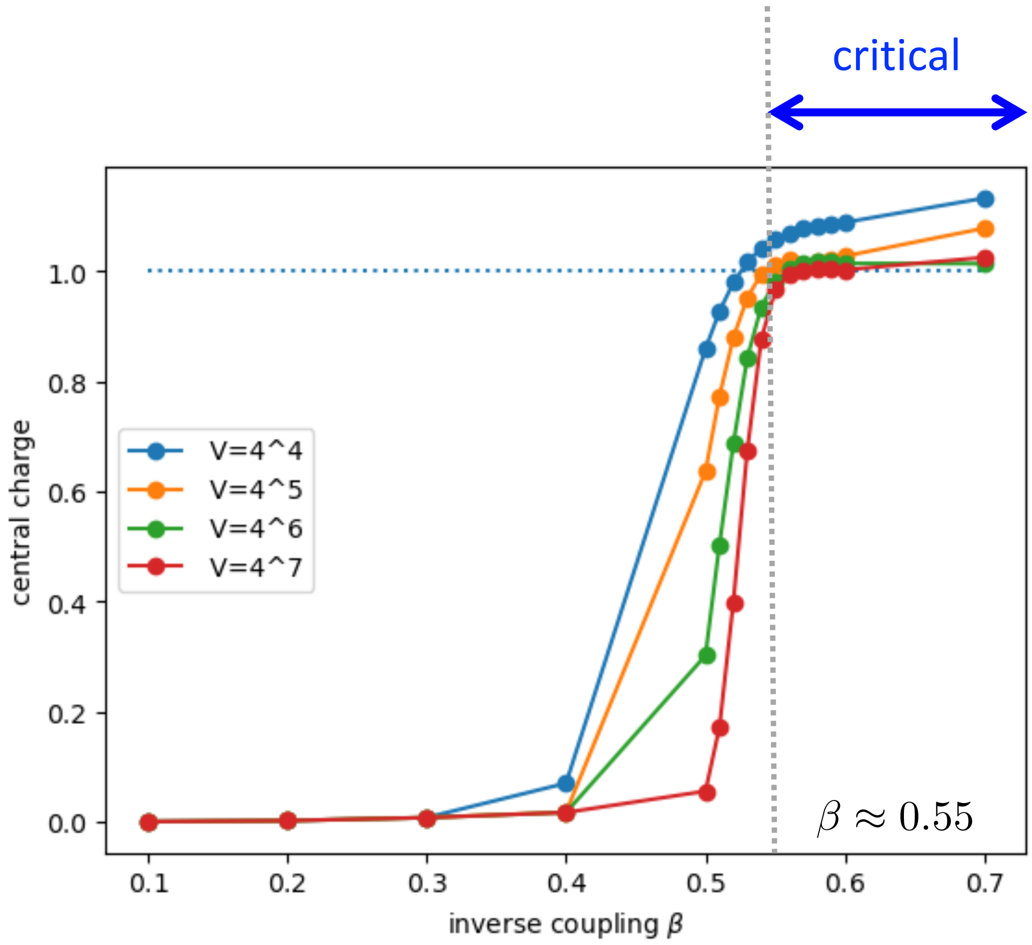
scaling dimension

$$x_i = h_i + \bar{h}_i = \frac{1}{2\pi}\log\left(\frac{\lambda_0}{\lambda_i}\right)$$

conformal weight

Central charge

critical $c = 1 \Leftrightarrow$ free boson CFT



$\theta = \pi$

$N_A = 120$

$N_z = 224$

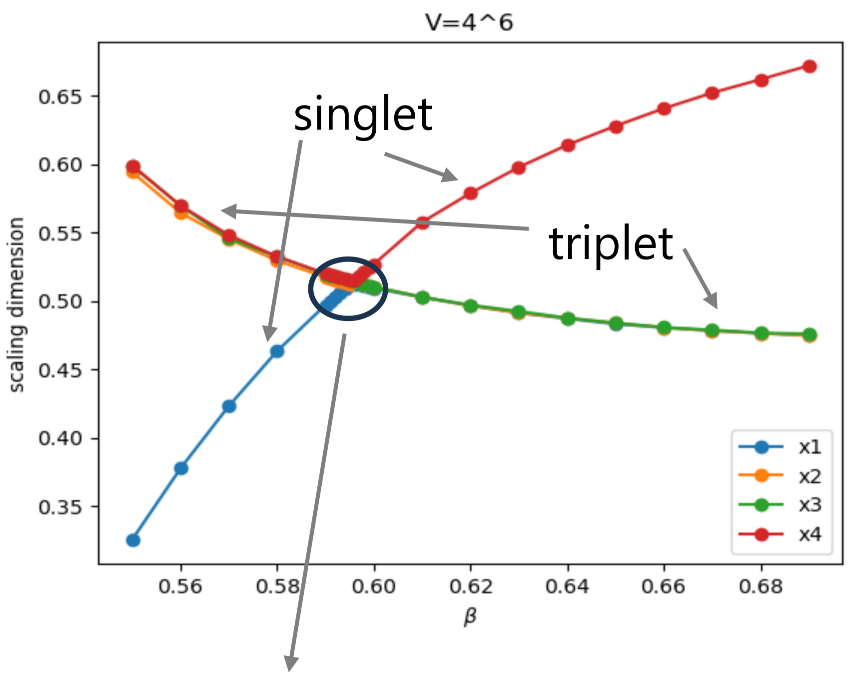
$\chi = 128$

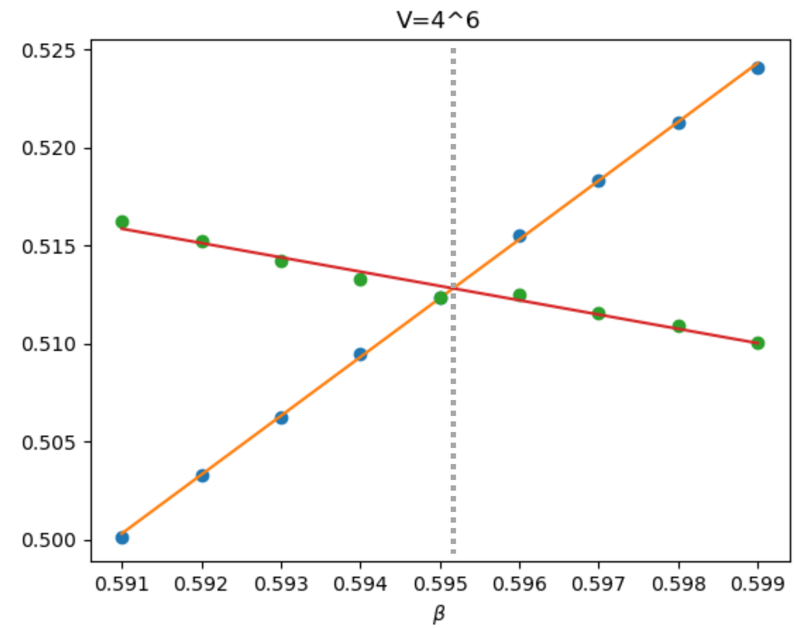
Bond weighted TRG

$k = -0.5$

Scaling dimensions

$$x_i = h_i + \bar{h}_i = \frac{1}{2\pi} \log \left(\frac{\lambda_0}{\lambda_i} \right) \qquad x_1, x_2, x_3, x_4$$





$$\beta_{k=1 SU(2)} \approx 0.595$$

quartet

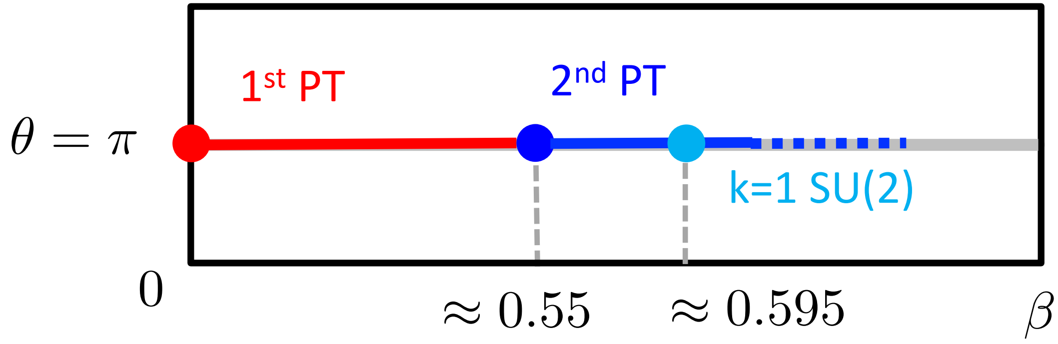
$$x_1 = x_2 = x_3 = x_4 \approx \frac{1}{2}$$

$\Leftrightarrow$ k=1 SU(2) WZNW

Wess+Zumino 1971, Novikov 1981, Witten 1984

Summary

- We have analyzed phase structure of 2d CP(1)+θ using tensor network
- Two improvements
  - Better (more precise) initial tensor
  - New analysis using CFT : central charge and scaling dimensions
- Critical point region starts from $\beta \approx 0.55$ and k=1 SU(2) symmetry is observed at $\beta \approx 0.595$



German-Japanese Workshop 202? in Kanazawa

If our proposal is accepted!!!

I need your help!!!

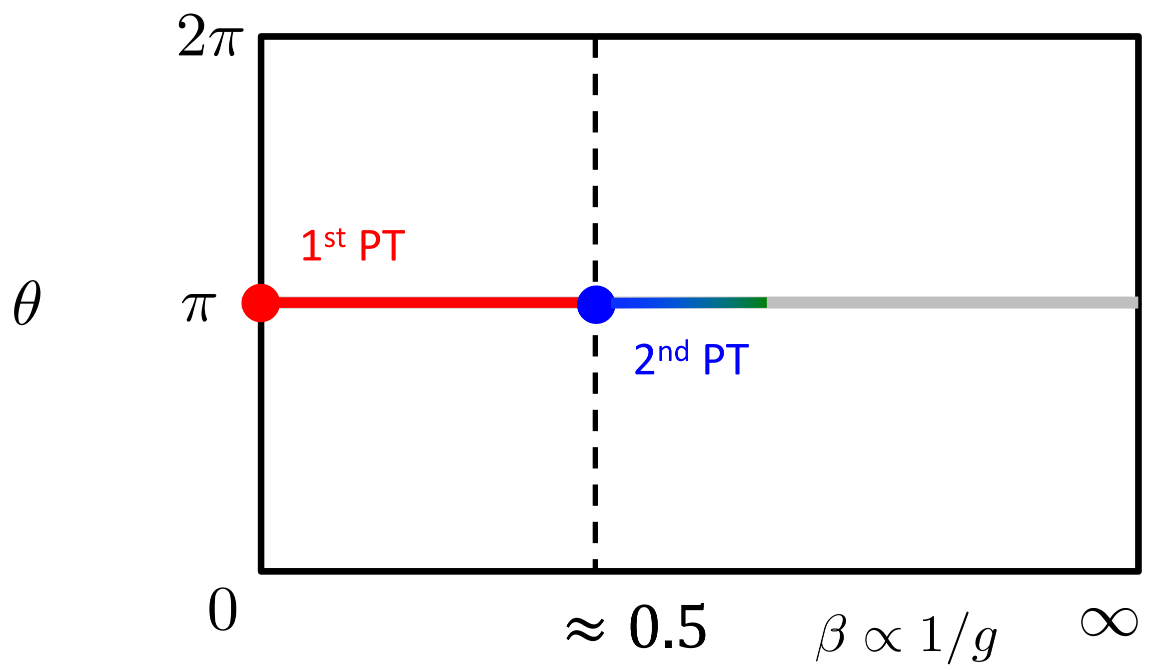
Future

- Mass gap scaling
- Continuum limit?

Back up

Previous study of phase structure

Kawauchi+ST 2018 Tensor network



Critical region is observed but the universality class is not identified

c=1, k=1 WZNW

scaling dimension for c=1 free boson CFT

$$x_i = x_{M,N} = \frac{M^2}{4R^2} + R^2 N^2, \qquad M, N \in \mathbb{Z}$$

R : compact radius

for $R = \sqrt{\frac{1}{2}}$ ⇒ SU(2) symmetry appears ⇒ k=1 WZNW universality class

Ginsparg 1988

$$x_{\pm 1,0} = x_{0,\pm} = \frac{1}{2}$$ ⇒ quartet

X

parameter

$N_z = 226,\ \ N_A = 120,$
$Dc = 128, \theta = \pi$

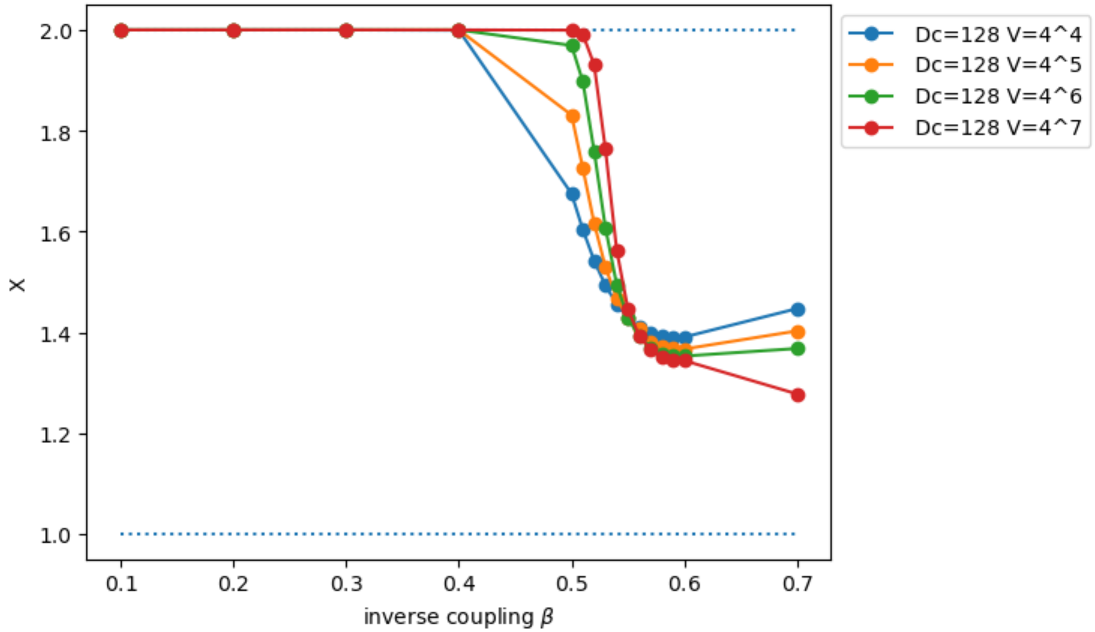
bond-weight TRG, $\mathrm{k} = -1/2$

D. Adachi, T. Okubo, and S. Todo,
Phys. Rev. B 105, L060402(2022)

$$X = \frac{(\sum_i T_{ii})^2}{\sum_{ij} T_{ij} T_{ji}} = \frac{(\sum_i \exp[-E_i T])^2}{\sum_i \exp[-2E_i T]}$$

$T \to \infty,$
$E_0 = E_1$のとき$X = 2$

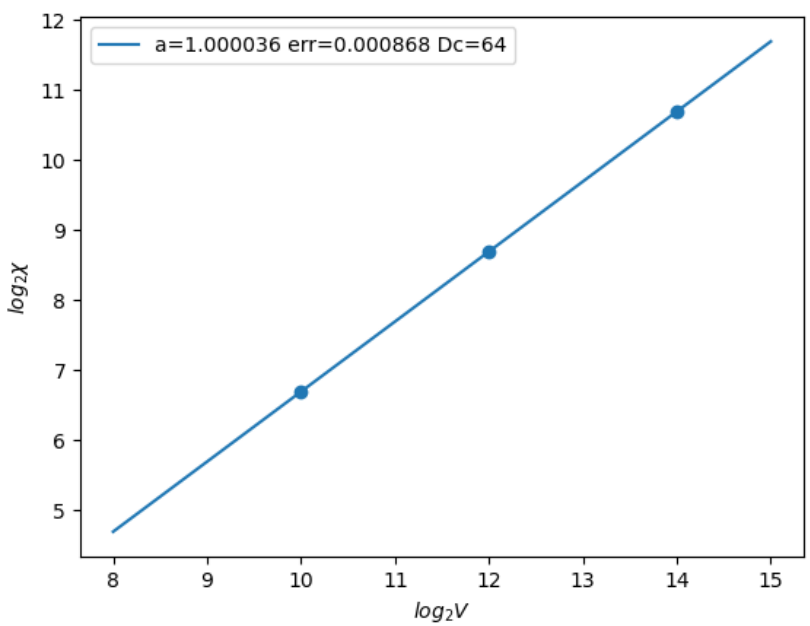
1st PT in $\beta = 0 - 0.5$



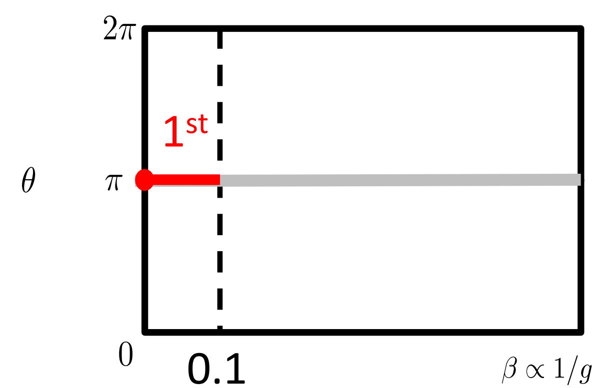
Nature of transition at $\beta = 0.1$

Kawauchi+ST 2017

$$\chi = \frac{1}{V}\frac{\partial^2}{\partial\theta^2}\log \mathcal{Z}(\theta)|_{\theta=\pi} \propto V^a \qquad \begin{cases} \text{1st} & : a = 1 \\ \text{2nd} & : a = \gamma/\nu < 1 \end{cases}$$



For $\beta = 0.1$
$a = 1.0000(9) \Rightarrow$ 1st PT



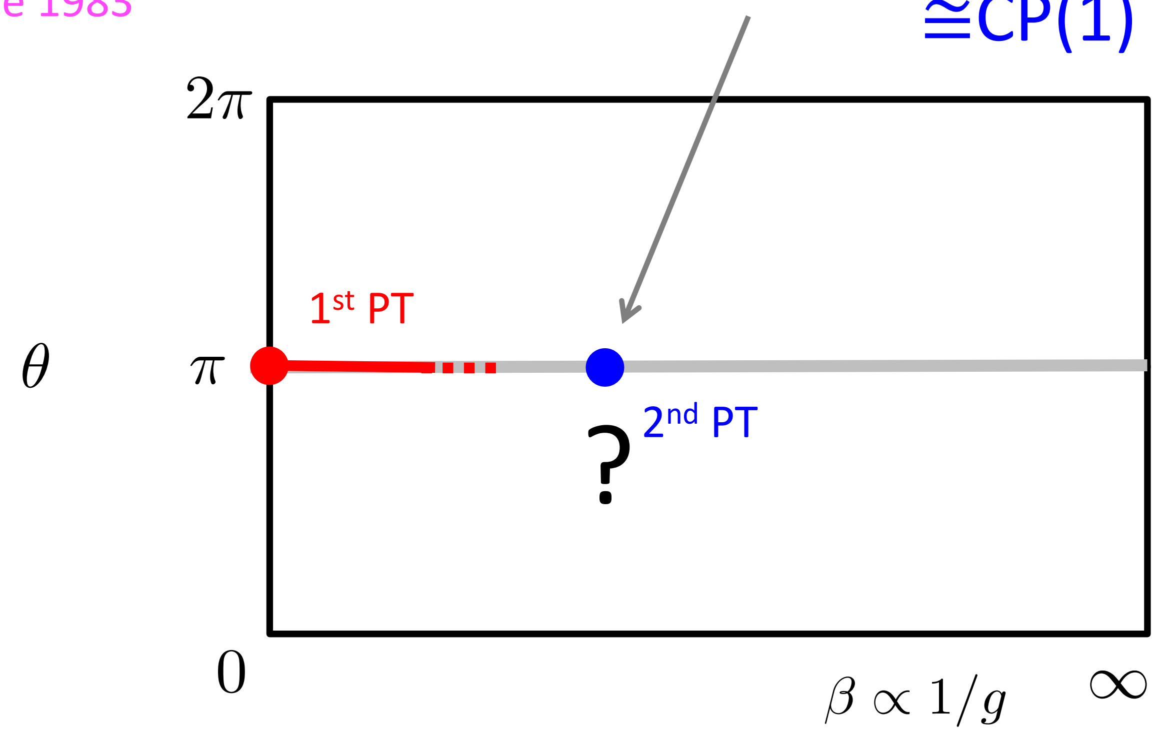
$D_{cut} = 64$, TRG for coarse-graining

Previous study of phase structure

Haldane's conjecture : mass gap of O(3) vanishes at θ=π

Haldane 1983

≅CP(1)



Bietenholtz et al. 1995, Wiese et al. 2012, de Forcrand et al. 2012, Azcoiti et al. 2012, Alles et al. 2014

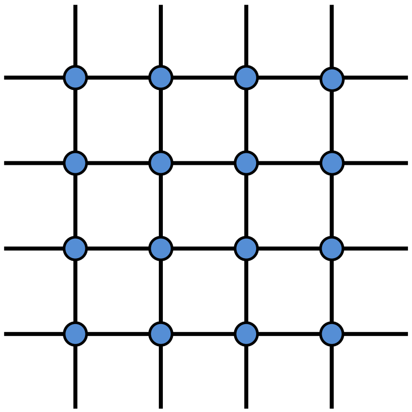
O(3) + θ was intensively studied by MC and Haldane's conjecture is confirmed

And universality class is consistent with k=1 WZNW model

Wess & Zumino 1971, Novikov 1981, Witten 1984

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



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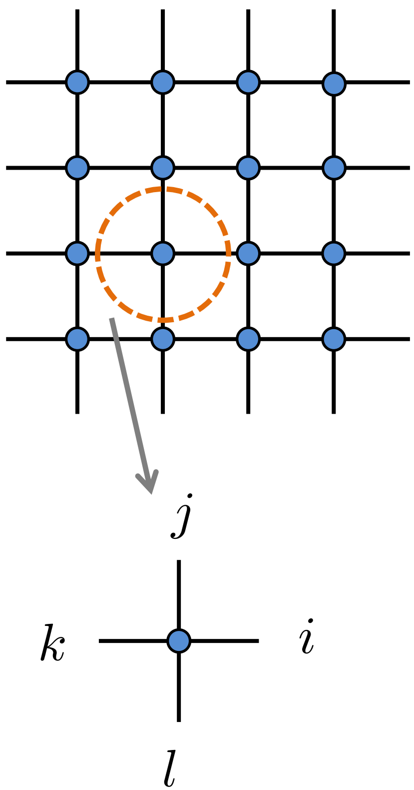


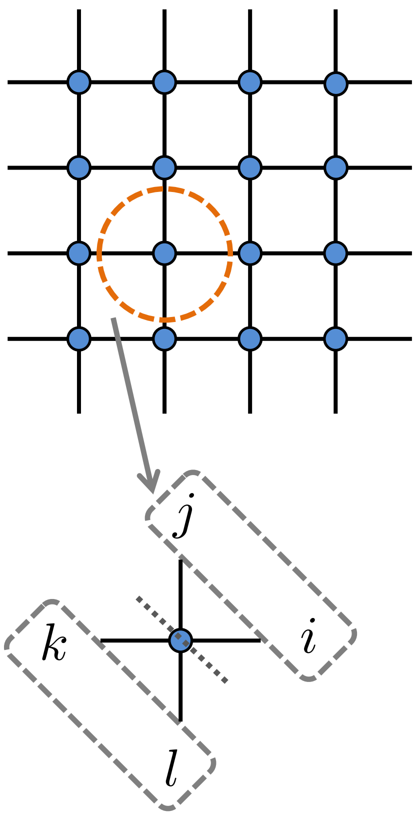
Bond dimension

$$1 \leq i, j, \ldots \leq \chi$$

$$\Leftrightarrow \quad T_{ijkl}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)



$$1 \leq i, j, \ldots \leq \chi$$

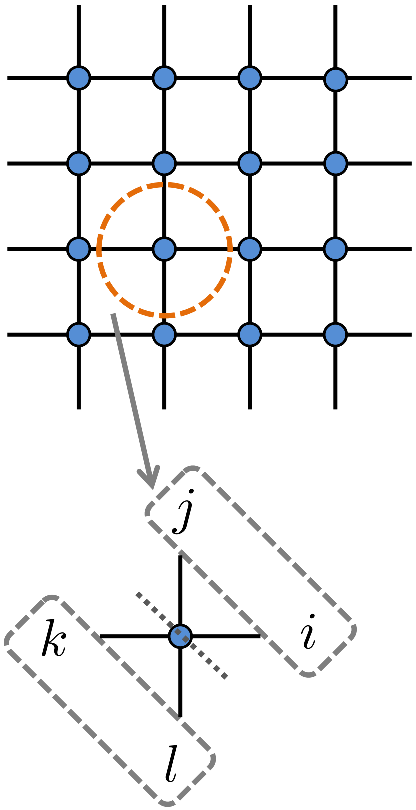
$\chi^2 \times \chi^2$ matrix

$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)

$$M \in \mathbb{C}^{\chi^2 \times \chi^2}$$



Singular Value Decomposition(SVD)

$$M_{ab} = \sum_m U_{am} \sigma_m (V^\dagger)_{mb}$$

unitary matrix

$$\sigma_1 \geq \sigma_2 \geq \ldots \geq 0$$

: singular value（non-negative）

$$\Leftrightarrow \quad T_{ijkl} = M_{(ij)(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)

$M \in \mathbb{C}^{\chi^2 \times \chi^2}$ ⇒ TN is sign-problem-free

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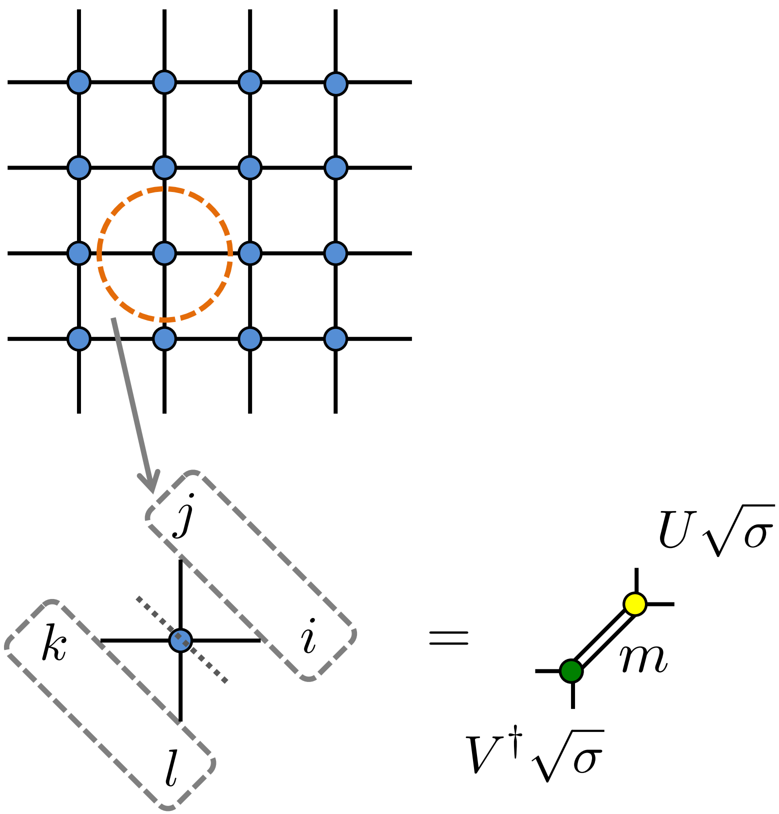

$\Leftrightarrow$

$$T_{ijkl} = M_{(ij)(kl)}$$

$$\overset{\text{SVD}}{=} \sum_{m=1}^{\chi^2} U_{(ij)m} \sigma_m V^\dagger_{m(kl)}$$

Coarse-graining

Tensor renormalization group (TRG)
PRL99,120601(2007)

$M \in \mathbb{C}^{\chi^2 \times \chi^2}$ ⇒ TN is sign-problem-free

Singular Value Decomposition(SVD)

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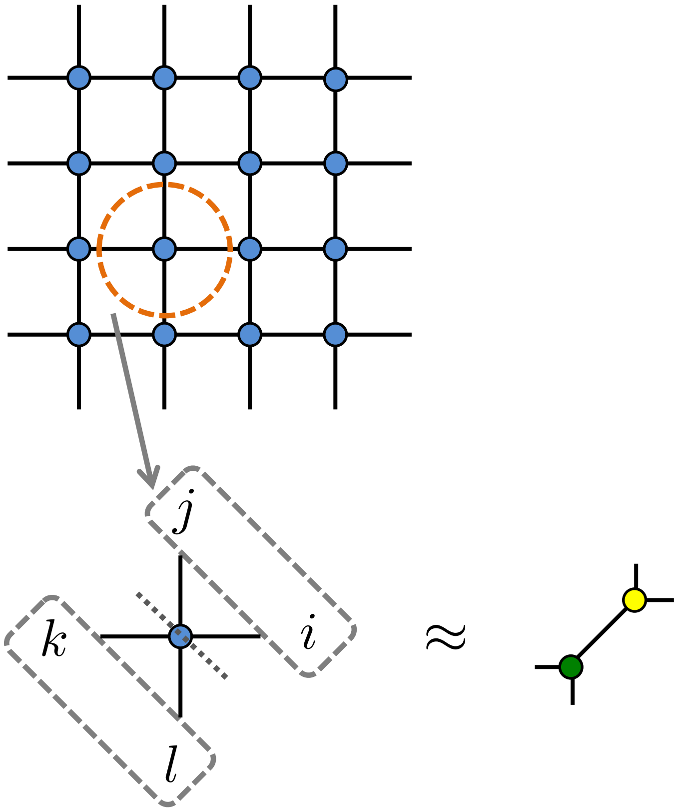
: singular value (non-negative)



$$T_{ijkl} = M_{(ij)(kl)}$$

truncation

$$\approx \sum_{m=1}^{\chi} U_{(ij)m} \sigma_m V^\dagger_{m(kl)}$$

truncation of SVD = information compression

Tensor network rep. of Z

depends on property of field and interaction

- Scalar field (non-compact)
  - Orthonormal basis expansion
    Shimizu mod.phys.lett. A27,1250035(2012), Lay & Rundnick PRL88,057203(2002)
  - Gauss Hermite quadrature Sakai et al., JHEP03(2018)141

- Gauge field (compact : SU(N), CP(N) etc.)
  - Character expansion : maintain symmetry, better convergence Meurice et al., PRD88,056005(2013)

- Fermion field (Dirac/Majorana)
  Shimizu & Kuramashi PRD90,014508(2014), ST & Yoshimura PTEP(2015)043B01
  - Grassmann number $\theta^2=0$ -> finite sum

$$e^{\phi\theta} = 1 + \phi\theta = \sum_{n=0}^{1} (\phi\theta)^n$$

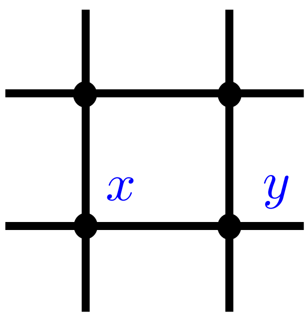
In principle, we can treat any fields

Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

Nearest Neighbor



Tensor network rep. for 2D Ising

e.g. 2D Ising model

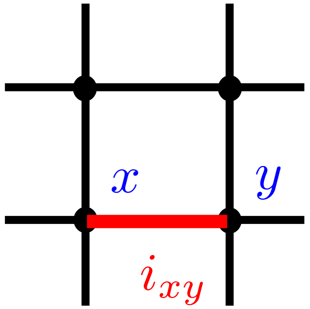
$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

$$= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}$$

V = # of lattice sites

$$\begin{aligned} \exp(\beta s_x s_y) &= \cosh(\beta s_x s_y) + \sinh(\beta s_x s_y) \\ &= \cosh\beta + s_x s_y \sinh\beta \\ &= \cosh\beta(1 + s_x s_y \tanh\beta) \\ &= \cosh\beta \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \end{aligned}$$

$$s_x = \pm 1$$



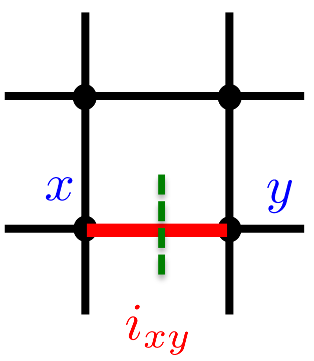
new d.o.f.

bond variable

Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\begin{aligned}\mathcal{Z} &= \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y) \\ &= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \\ &= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x \sqrt{\tanh\beta} \cdot s_y \sqrt{\tanh\beta})^{i_{xy}}\end{aligned}$$



Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y)$$

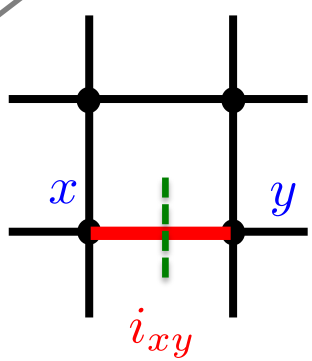
$$= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}}$$

$$= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x \sqrt{\tanh\beta} \cdot s_y \sqrt{\tanh\beta})^{i_{xy}}$$

$$= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (s_x \sqrt{\tanh\beta})^{i_{xy}} (s_x \sqrt{\tanh\beta})^{i_{xz}} (s_x \sqrt{\tanh\beta})^{i_{xw}} (s_x \sqrt{\tanh\beta})^{i_{xv}}$$

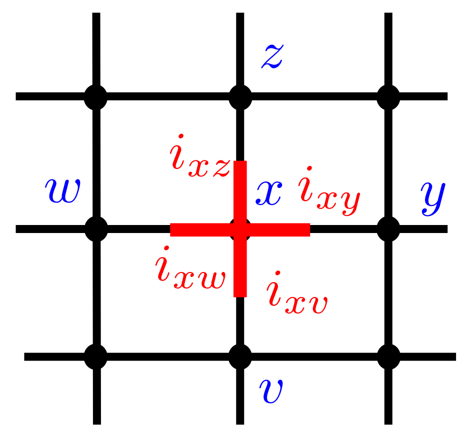
$$\prod_{<x,y>}$$

bond



$$\prod_{x}$$

lattice point



Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$
\begin{aligned}
\mathcal{Z} &= \sum_{\{s\}} \exp\left(\sum_{<x,y>} \beta s_x s_y\right) = \sum_{\{s\}} \prod_{<x,y>} \exp(\beta s_x s_y) \\
&= (\cosh\beta)^{2V} \sum_{\{s\}} \prod_{<x,y>} \sum_{i_{xy}=0}^{1} (s_x s_y \tanh\beta)^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{<x,y>} (s_x\sqrt{\tanh\beta} \cdot s_y\sqrt{\tanh\beta})^{i_{xy}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (s_x\sqrt{\tanh\beta})^{i_{xy}} (s_x\sqrt{\tanh\beta})^{i_{xz}} (s_x\sqrt{\tanh\beta})^{i_{xw}} (s_x\sqrt{\tanh\beta})^{i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \sum_{\{s\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} (\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \sum_{s_x=\pm 1} s_x^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} \\
&= (\cosh\beta)^{2V} \sum_{\{i\}} \prod_{x} \boxed{(\sqrt{\tanh\beta})^{i_{xy}+i_{xz}+i_{xw}+i_{xv}} 2\delta(\mathrm{mod}(i_{xy}+i_{xz}+i_{xw}+i_{xv}, 2))}
\end{aligned}
$$

summation is done

$$= T_{i_{xy} i_{xz} i_{xw} i_{xv}}$$

new d.o.f. : index of tensor

Tensor network rep. for 2D Ising

e.g. 2D Ising model

$$\mathcal{Z} = 2^V (\cosh\beta)^{2V} \sum_{..,i,j,k,l,m,n,o,..} \cdots T_{ijkl} T_{mnio} \cdots$$

$$T_{ijkl} = (\sqrt{\tanh\beta})^{i+j+k+l} \delta(\mathrm{mod}(i+j+k+l), 2)$$

$$\begin{bmatrix} T_{0000} & T_{0001} & T_{0010} & T_{0011} \\ T_{0100} & T_{0101} & T_{0110} & T_{0111} \\ T_{1000} & T_{1001} & T_{1010} & T_{1011} \\ T_{1100} & T_{1101} & T_{1110} & T_{1111} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \tanh\beta \\ 0 & \tanh\beta & \tanh\beta & 0 \\ 0 & \tanh\beta & \tanh\beta & 0 \\ \tanh\beta & 0 & 0 & (\tanh\beta)^2 \end{bmatrix}$$

size and elements of tensor depend on a model

MC	TRG
Boltzmann weight is interpreted as probability	Tensor network rep. of partition function (no probability interpretation)
Importance sampling	Compression of tensor by SVD, Optimization
Statistical errors	Systematic errors (truncated SVD)
Sign problem may appear	No sign problem ∵ no probability
Critical slowing down	Efficiency of compression gets worse around criticality

↓

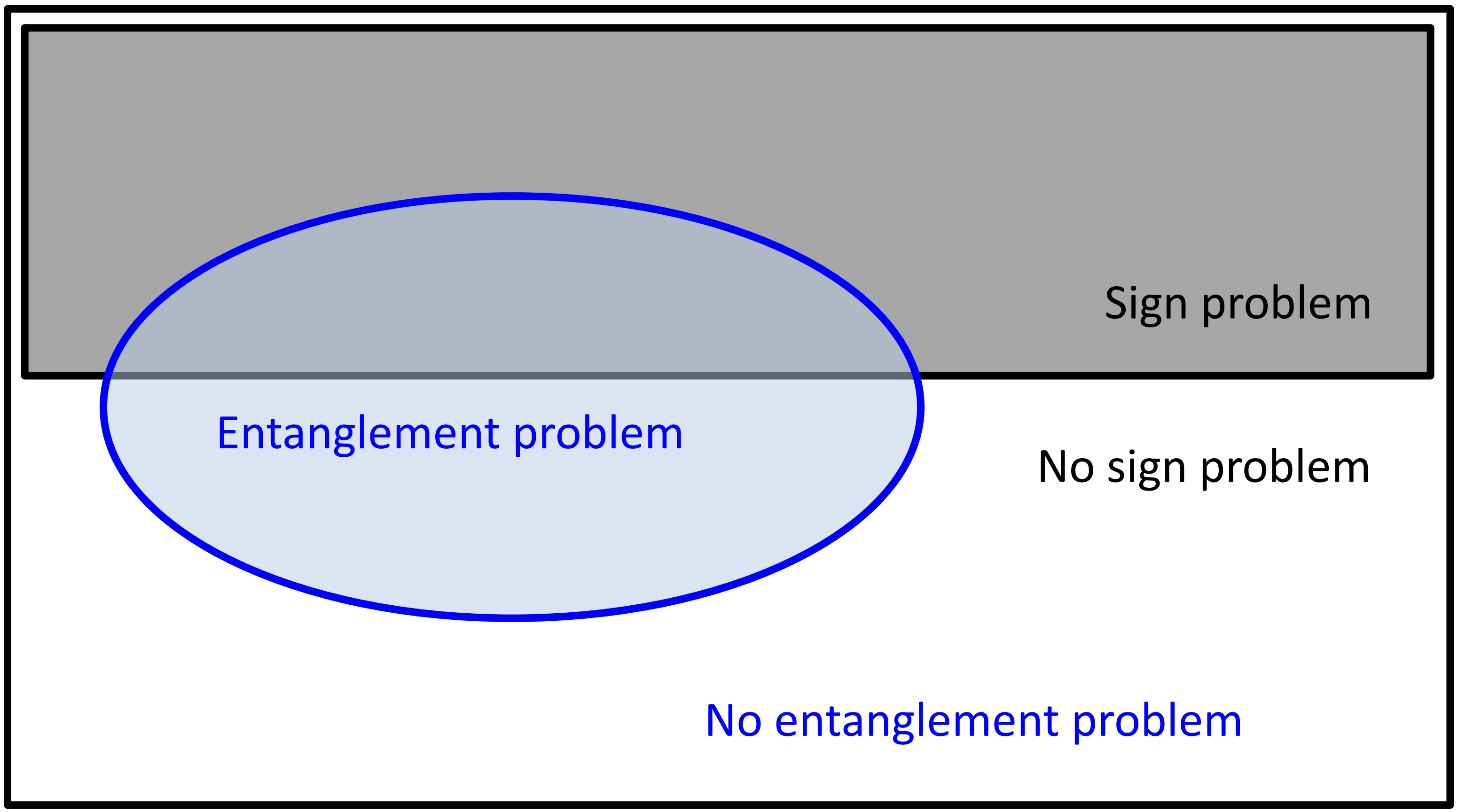
can be improved by TNR, Loop-TNR in 2D system

Evenbly & Vidal 2014, Gu et al., 2015

From point of view of MC

From point of view of TN

Space of theory

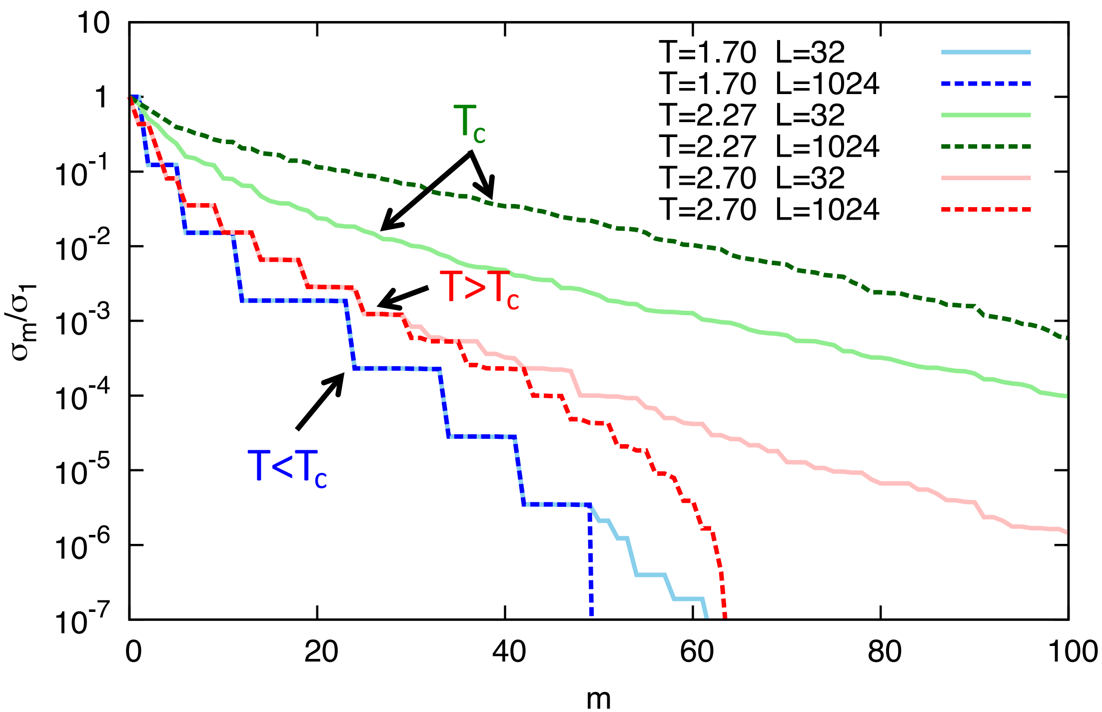


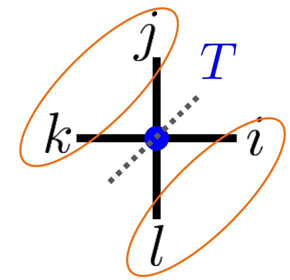
Is there a region where any method cannot access ?

Is entanglement problem NP-hard ?

Hierarchy of singular value

2D Ising model D_{cut}=32





$$T_{(kj)(il)} = \sum_m u_{(kj)m}\sigma_m v_{m(il)}$$

$$L = 2^{\#\text{RG step}}$$

$$T_c = 2/[\ln(1+\sqrt{2})] = 2.269...$$

- Off criticality: good hierarchy (small S)
- Near criticality: hierarchy gets worse (large S)

like critical slowing down in MC

Tensor network renormalization (TNR) Evenbly&Vidal 2014 can help the situation

Renormalization group

$$H(K;\{s\}) = \sum_i K_i \mathcal{O}_i(\{s\})$$

s : spins

$$\mathcal{Z} = \sum_{\{s\}} e^{-\beta H}$$

Block spin transf., Migdal-Kadanoff RG

$$H = \sum_i K_i \mathcal{O}_i \xrightarrow{\mathcal{R}} H' = \sum_i K'_i \mathcal{O}'_i$$

$$K' = \mathcal{R}_K(K)$$

$$K^* = \mathcal{R}_K(K^*)$$: fixed point

target : critical exponent etc

TRG

$$T_{ijkl}(K)$$

i, j, k, l : indexes

$$\mathcal{Z} = \sum_{i,j,k,l,\ldots} \prod T_{ijkl} \cdots$$

SVD + contraction

$$\sum_{i,j,k,l,\ldots} \prod T_{ijkl} \cdots \xrightarrow{\mathcal{R}} \sum_{i,j,k,l,\ldots} \prod T'_{ijkl} \cdots$$

$$T' = \mathcal{R}_T(T)$$

$$T^* = \mathcal{R}_T(T^*)$$: fixed point tensor

target : partition function etc

Kadanoff et al, Rev.Mod.Phys.86,647(2014)

SVD

For simplicity

rank-k $m \times n$ real matrix A ($m \geq n \geq k$) is given by

$$A = U \Lambda V^T$$ (full SVD)

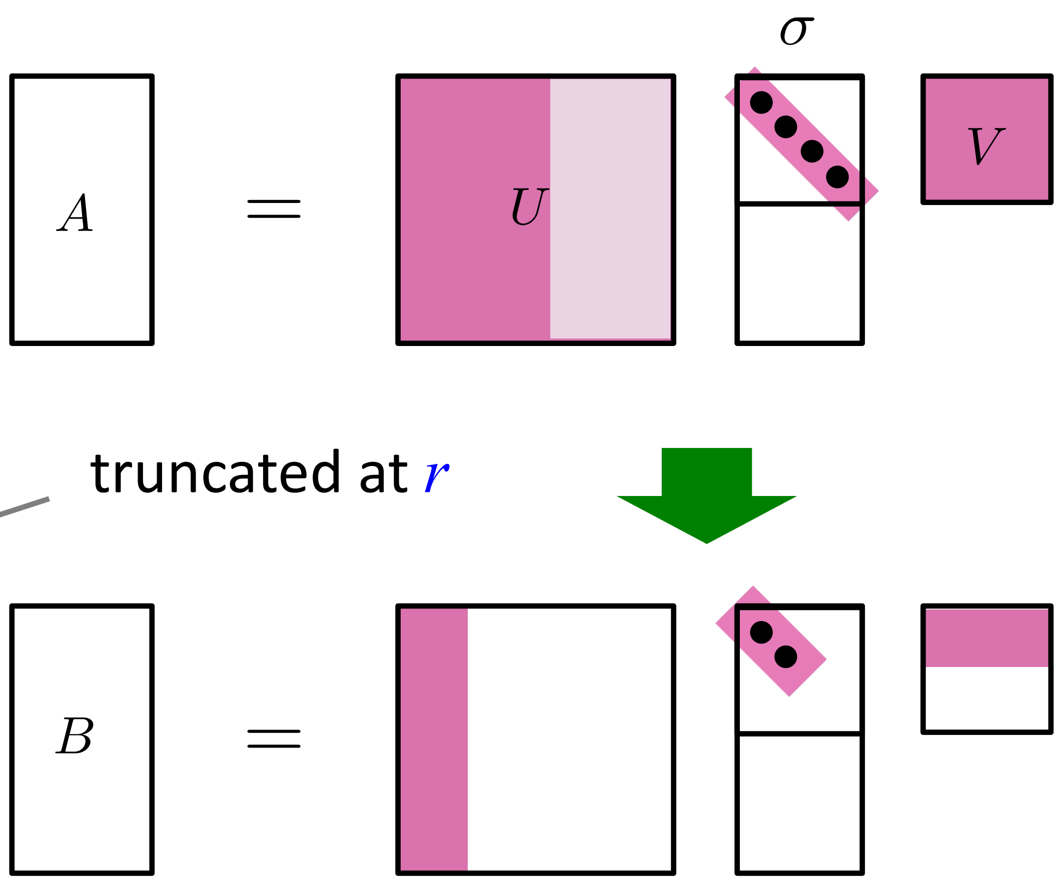
- U : $m \times m$ orthonormal matrix : $U=(u_1, u_2,..., u_m)$, $U^T U = U U^T = I_m$
- V : $n \times n$ orthonormal matrix : $V=(v_1, v_2,..., v_n)$, $V^T V = V V^T = I_n$
- Λ : $m \times n$ diagonal matrix : $\Lambda = \mathrm{diag}(\sigma_1, \sigma_2,..., \sigma_n)$

where $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k > \sigma_{k+1} = \ldots = \sigma_n = 0$)

$$A = \sum_{l=1}^{k} \sigma_l u_l v_l^T$$ Decomposition using rank-1 tensor

Best approximation of A ?

$$A = \sum_{l=1}^{k} \sigma_l u_l v_l^T$$



$$B = \sum_{l=1}^{r} \sigma_l u_l v_l^T$$