“Tensor renormalization group study of (1+1)-dimensional U(1) gauge-Higgs model at $\theta = \pi$ with Lüscher’s admissibility condition”

S. Akiyama and YK, JHEP09(2024)086

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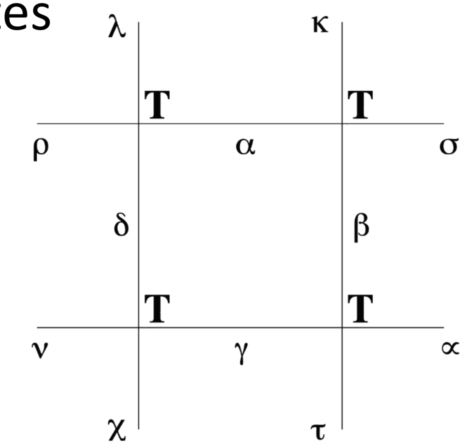
Plan of talk

- Brief Introduction to Tensor Renormalization Group(TRG)
- Current Status of TRG Application to QFTs
- (1+1)-dimensional U(1) gauge-Higgs model w/ θ-term
  - Pure U(1) gauge theory w/ $M \to \infty$
  - Determination of critical endpoint M_c
- Summary

Tensor Renormalization Group (TRG)

Levin-Nave
PRL99(2007)120601

Explain the algorithm with 2D Ising model with N sites



Hamiltonian $H = \sum_{\langle i,j \rangle} s_i s_j \qquad s_i \pm 1$

Partition Function $Z = \sum_{\{s_i\}} \exp\left(-\beta H\right)$

$$= \sum_{\alpha,\beta,\gamma,\delta,\cdots=1}^{2} T_{\alpha,\lambda,\rho,\delta} T_{\sigma,\kappa,\alpha,\beta} T_{\mu,\beta,\gamma,\tau} T_{\gamma,\delta,\nu,\chi} \cdots$$

Tensor Network representation

Details of model are specified in initial tensor
The algorithmic procedure is independent of models

Of course, direct contraction is impossible for large N even with current fastest supercomputer

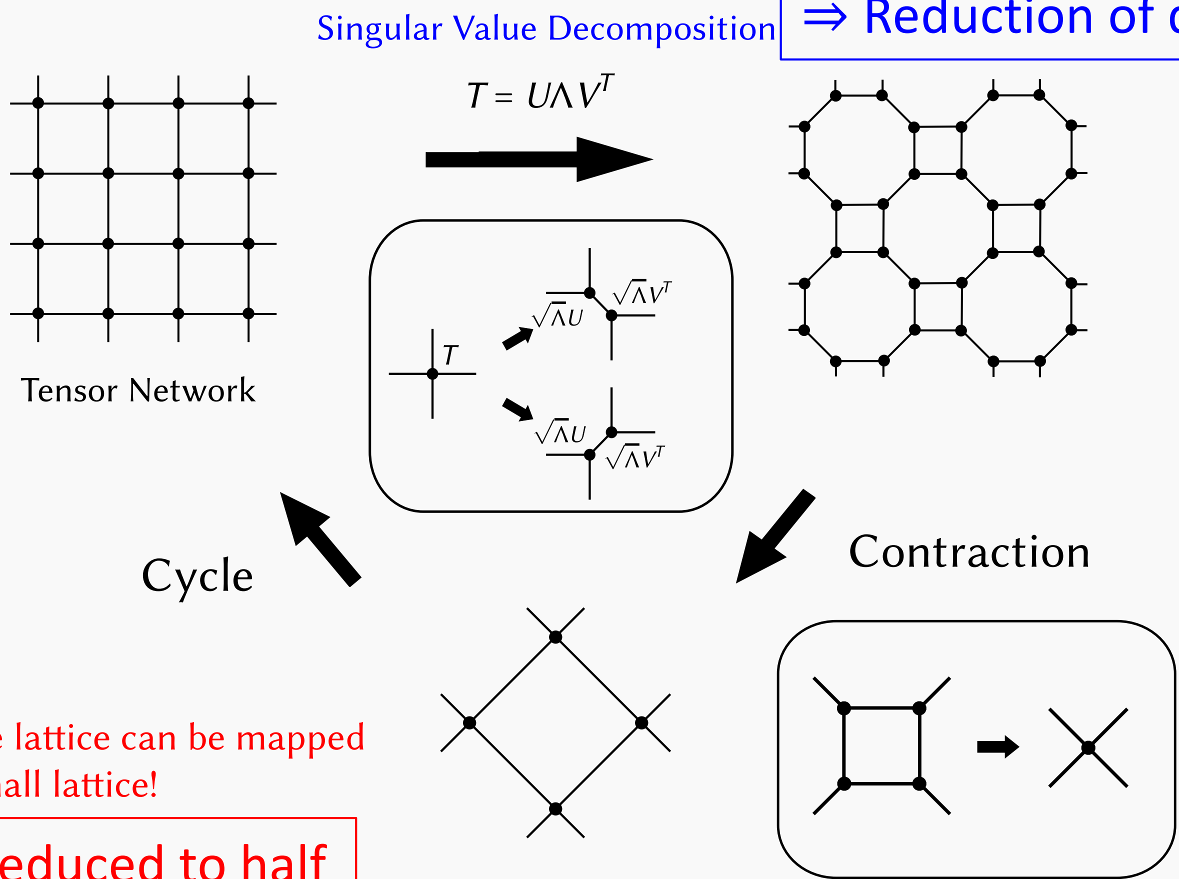
⇒ How to evaluate the partition function?

Schematic View of TRG Algorithm

1. Singular Value Decomposition of local tensor T
2. Contraction of old tensor indices (coarse-graining)
3. Repeat the iteration

Tensor Network Renormalization Group

Keep largest D_{cut} components
⇒ Reduction of d.o.f.



Tensors w/ new indices

Large lattice can be mapped to small lattice!

#sites are reduced to half

TRG vs Monte Carlo

Monte Carlo
stochastic

antithetical principles

TRG
deterministic

<u>Advantages of TRG</u>

- Free from sign problem/complex action problem in MC method

$$Z = \int \mathcal{D}\phi \ \exp(-S_{\rm Re}[\phi] + iS_{\rm Im}[\phi])$$

- Computational cost for L^D system size $\propto D \times \log(L)$
- Direct manipulation of Grassmann numbers
- Direct evaluation of partition function Z (density matrix ρ) itself

Applications in particle physics :

Finite density QCD, QFTs w/ θ-term, Lattice SUSY etc.

Also, in condensed matter physics

Hubbard model (Mott transition, High Tc superconductivity) etc.

TRG Approaches to QFTs (1)

w/ sign problem

<u>2d models</u>

Real ϕ^4 theory :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP05(2019)184

Complex ϕ^4 theory w/ $\mu\neq0$:

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP02(2020)161

U(1) gauge theory w/ θ-term :

YK-Yoshimura, JHEP04(2020)089

Schwinger(2d QED), Schwinger w/ θ-term :

Shimizu-YK, PRD90(2014)014508, 074503, PRD97(2018)034502

N=1 Wess-Zumino model (SUSY) :

Kadoh-YK-Nakamura-Sakai-Takeda-Yoshimura, JHEP03(2018)141

SU(2) principal chiral model w/ $\mu\neq0$:

Luo-YK, PRD107(2023)094509

O(3) nonlinear σ model w/ $\mu\neq0$ and μ=0 :

Luo-YK, JHEP03(2024)020, arXiv:2406.08865

Application to various models w/ sign problem,
Development of calculational methods for scalar, fermion and gauge fields

TRG Approaches to QFTs (2)

w/ sign problem

3d models

Z_2 gauge-Higgs model w/ $\mu \neq 0$: Akiyama-YK,JHEP05(2022)102
Real ϕ^4 theory : Akiyama-YK-Yoshimura, PRD104(2021)034507
Z_2gauge theory at $T \neq 0$: YK-Yoshimura, JHEP08(2019)023

4d models

Ising model : Akiyama-YK-Yamashita-Yoshimura, PRD100(2019)054510
Complex ϕ^4 theory w/ $\mu \neq 0$:
Akiyama-Kadoh-YK-Yamashita-Yoshimura, JHEP09(2020)177
NJL model w/ $\mu \neq 0$:
Akiyama-YK-Yamashita-Yoshimura, JHEP01(2021)121
Real ϕ^4 theory : Akiyama-YK-Yoshimura, PRD104(2021)034507
Z_2 gauge-Higgs model w/ $\mu \neq 0$: Akiyama-YK, JHEP05(2022)102
Z_3 gauge-Higgs model w/ $\mu \neq 0$: Akiyama-YK, JHEP10(2023)077

⇒ Research target is shifting from 2d models to 4d ones

TRG Approaches to QFTs (3)

w/ sign problem

Condensed matter physics

Similarity btw Hubbard models and NJL ones

Action consisting of hopping terms and 4-fermi interaction term

$$S=\sum_{n\in\Lambda_{d+1}}\epsilon\left\{\bar{\psi}(n)\left(\frac{\psi(n+\hat{\tau})-\psi(n)}{\epsilon}\right)-t\sum_{\sigma=1}^{d}(\bar{\psi}(n+\hat{\sigma})\psi(n)+\bar{\psi}(n)\psi(n+\hat{\sigma}))+\frac{U}{2}(\bar{\psi}(n)\psi(n))^2-\mu\bar{\psi}(n)\psi(n)\right\}$$

First principle calculation at finite density

- (1+1)d Hubbard model : Akiyama-YK, PRD104(2021)014504
- (2+1)d Hubbard model : Akiyama-YK-Yamashita, PTEP2022(2022)023I01

(1+1)d U(1) gauge Higgs model w/ θ-term imposing Lüscher's admissibility condition

Simultaneous solution to complex action problem and topological freezing problem

cf. dual simulation w/o Lüscher's admissibility condition

Gattringer+, PRD92(2015)114508, NPB935(2018)344

Action and expected phase diagram

(1+1)d lattice U(1) gauge–Higgs model w/ θ–term

$$Z = \prod_{n,\nu} \int_{-\pi}^{\pi} \frac{\mathrm{d}\vartheta_\nu(n)}{2\pi} \prod_n \int_{\mathbb{C}} \frac{\mathrm{d}\phi(n)}{2\pi} \exp(-S) \qquad S = \beta S_g + S_h + S_\theta$$

$$S_g = \begin{cases} \sum_n \frac{1 - \mathrm{Re}P_{12}(n)}{1 - [1 - \mathrm{Re}P_{12}(n)]/\epsilon} & \text{if "admissible"} \\ \infty & \text{otherwise} \end{cases}$$

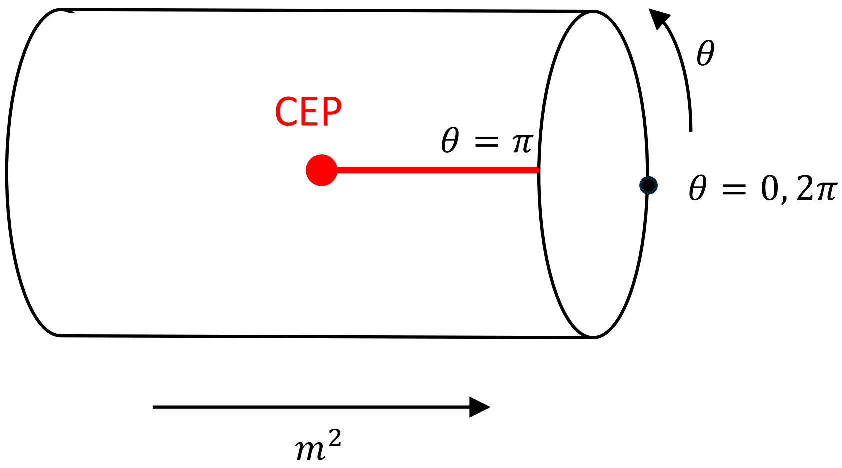
Lüscher, NPB549(1999)295

P_{12} is plaquette value, link variable: $U_\nu(n) = \mathrm{e}^{\mathrm{i}\vartheta_\nu(n)} \quad \vartheta_\nu(n) \in [-\pi, \pi]$

$$S_h = -\sum_n \sum_\nu [\phi^*(n) U_\nu(n) \phi(n+\hat{\nu}) + \phi^*(n+\hat{\nu}) U_\nu^*(n) \phi(n)]$$
$$+ M \sum_n |\phi(n)|^2 + \lambda \sum_n |\phi(n)|^4 .$$

$$M = m^2 + 4$$

$$S_\theta = -\frac{\mathrm{i}\theta}{2\pi} \sum_n \ln P_{12}(n)$$



Reparametrization of Higgs field

Complex-valued Higgs field is reparameteized by $\phi(n) = \sqrt{l(n)}e^{i\varphi(n)}$

$$S_h = -\sum_n \sum_\nu 2\sqrt{\ell(n)\ell(n+\hat{\nu})}\cos\left[\varphi(n+\hat{\nu}) - \varphi(n) + \vartheta_\nu(n)\right] + \sum_n \left[M\ell(n) + \lambda\ell(n)^2\right]$$

Choose unitary gauge $\Longrightarrow$ eliminate $\varphi(n)$ variables

$$Z = \prod_{n,\nu}\int_{-\pi}^{\pi}\frac{\mathrm{d}\vartheta_\nu(n)}{2\pi}\prod_n\int_0^\infty\frac{\mathrm{d}\ell(n)}{2}\exp\left[-\beta S_g - S_h' - S_\theta\right]$$

$$S_h' = -\sum_n\sum_\nu 2\sqrt{\ell(n)\ell(n+\hat{\nu})}\cos\vartheta_\nu(n) + \sum_n\left[M\ell(n) + \lambda\ell(n)^2\right]$$

Discretization is necessary to construct tensor network representation

Integral over $l(n)$: Gauss–Laguerre quadrature

Integral over $\vartheta(n)$: Gauss–Legendre quadrature

Discretized Action w/ Gauss Quadrature

$$Z \simeq Z(K_g, K_h)$$

$$= \prod_{n,\nu} \sum_{\tilde{\vartheta}_\nu(n)\in D_g} \frac{w_{\tilde{\vartheta}_\nu(n)}}{2} \prod_n \sum_{\tilde{\ell}(n)\in D_h} \frac{w_{\tilde{\ell}(n)} \mathrm{e}^{\tilde{\ell}(n)}}{2} \exp\left[-\beta \tilde{S}_g - \tilde{S}'_h - \tilde{S}_\theta\right]$$

No. of sample points: (K_g, K_h), weights: $(w_{\tilde{\vartheta}_\nu}, w_{\tilde{l}})$

$$\tilde{S}_g = \begin{cases} \sum_n \frac{1-\cos\pi\left(\tilde{\vartheta}_1(n)+\tilde{\vartheta}_2(n+\hat{1})-\tilde{\vartheta}_1(n+\hat{2})-\tilde{\vartheta}_2(n)\right)}{1-\left[1-\cos\pi\left(\tilde{\vartheta}_1(n)+\tilde{\vartheta}_2(n+\hat{1})-\tilde{\vartheta}_1(n+\hat{2})-\tilde{\vartheta}_2(n)\right)\right]/\epsilon} & \text{if admissible} \\ \infty & \text{otherwise} \end{cases},$$

$$\tilde{S}'_h = -\sum_n \sum_\nu 2\sqrt{\tilde{\ell}(n)\tilde{\ell}(n+\hat{\nu})} \cos\pi\tilde{\vartheta}_\nu(n) + \sum_n \left[M\tilde{\ell}(n) + \lambda\tilde{\ell}(n)^2\right]$$

$$\tilde{S}_\theta = -\frac{\mathrm{i}\theta}{2\pi} \sum_n \ln\left[\mathrm{e}^{\mathrm{i}\pi\left(\tilde{\vartheta}_1(n)+\tilde{\vartheta}_2(n+\hat{1})-\tilde{\vartheta}_1(n+\hat{2})-\tilde{\vartheta}_2(n)\right)}\right]$$

Tensor network representation is constructed based on $Z(K_g, K_h)$

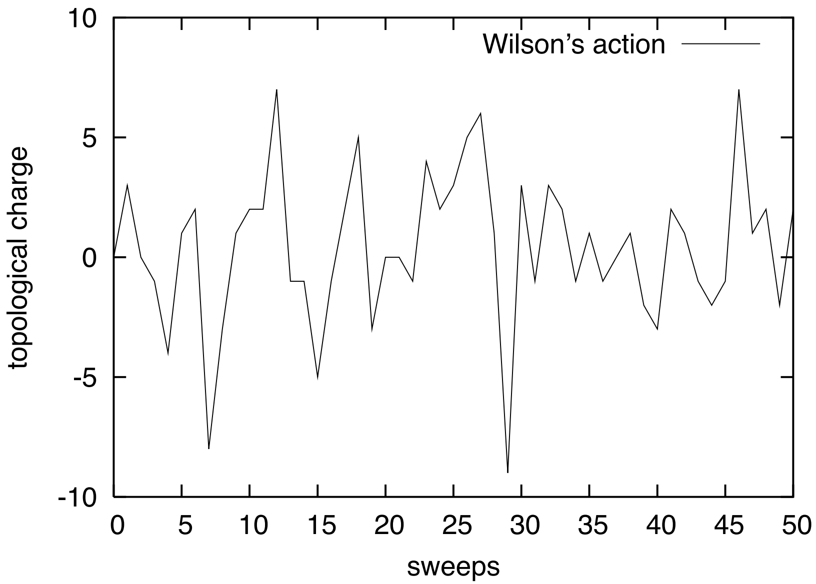
We employ Bond-Weighted TRG (BTRG) Adachi-Okubo-Todo, PRB105(2022)L0604020

Topological Freezing Problem

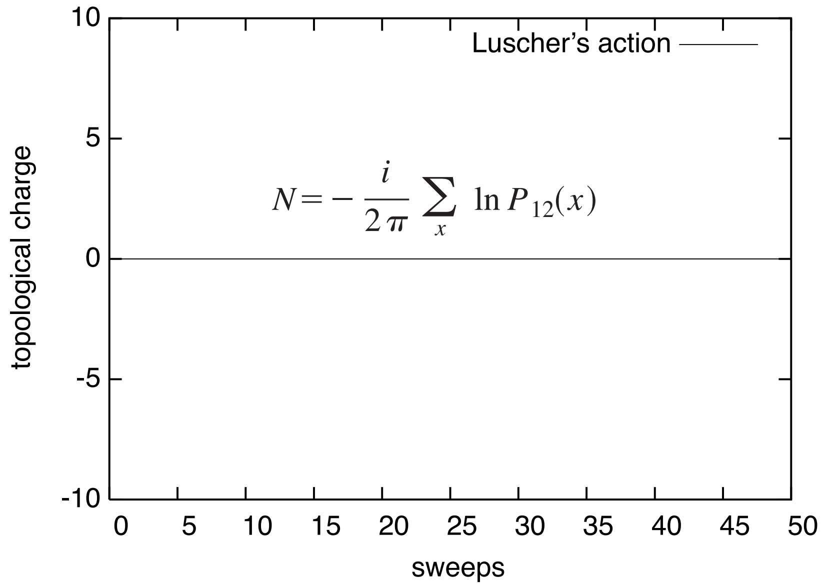
Hybrid Monte Carlo method for Lüscher action

Fukaya-Onogi, PRD63(2003)074503

pure U(1)@$\beta = 3.4$



pure U(1)@$\beta = 1.0, \epsilon = 1.0$



No topology change w/ Lüscher action@$\beta = 1.0, \epsilon = 1.0$

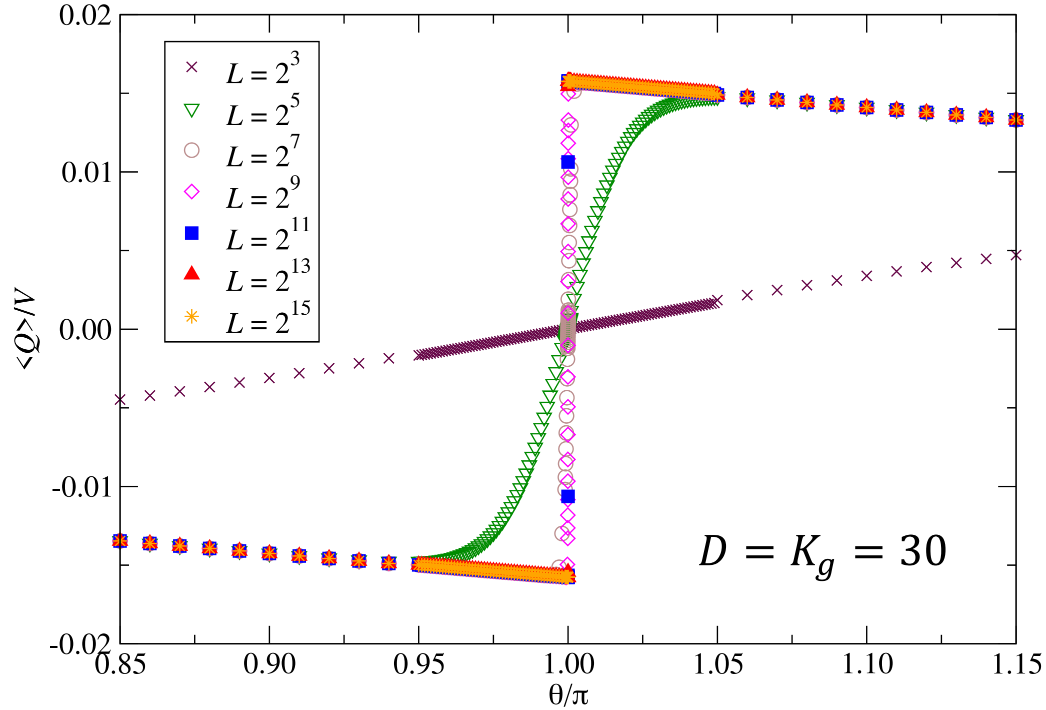
Note: Wilson action also has topological freezing problem at larger β

Pure U(1) Gauge Theory w/ θ-term (1)

Pure U(1) gauge = U(1) gauge−Higgs at $M\rightarrow\infty$

Parameter choice in this work: $\beta = 3.0$, $\epsilon = 1.0$

topological charge density: $\dfrac{\langle Q\rangle}{V} = -\dfrac{\mathrm{i}}{V}\dfrac{\partial \ln Z}{\partial\theta}$



Clear jump at $\theta = \pi$ w/ $L \geq 2^{13}$ ⇒ 1st order phase transition

Spontaneous Z_2 symmetry breaking

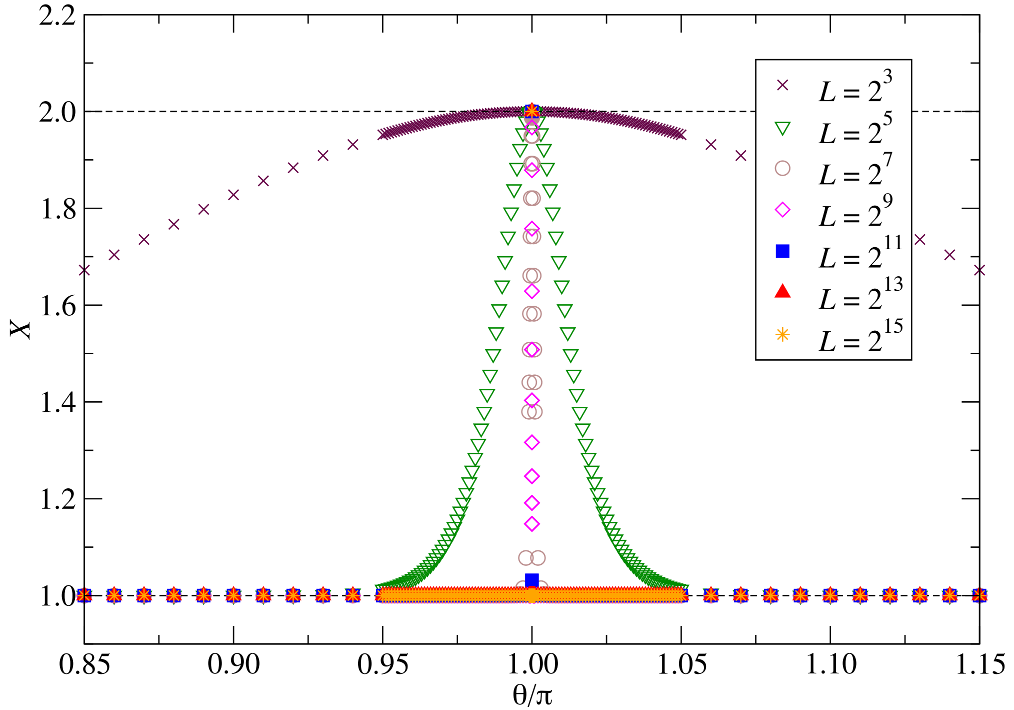
Pure U(1) Gauge Theory w/ θ-term (2)

Measurement of ground state degeneracy

Gu-Wen, PRB580(2009)155131

$$X = \frac{(\mathrm{Tr} A)^2}{\mathrm{Tr}\,(A^2)} \qquad A_{yy'} = \sum_x T_{xyxy'}$$

- X=1: symmetric
- X=2: broken

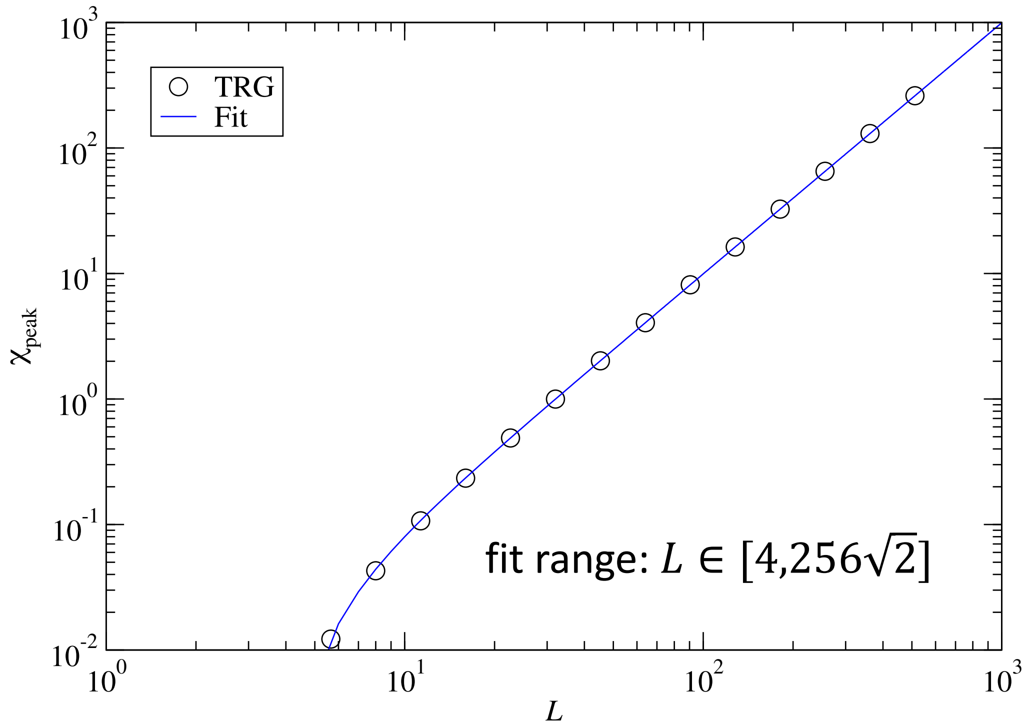


X=2 at $\theta = \pi$ w/ $L \geq 2^{13}$ ⇒ Spontaneous Z_2 symmetry breaking

Pure U(1) Gauge Theory w/ θ-term (3)

Finite size scaling analysis on topological susceptibility

$$\chi_Q = \frac{\partial}{\partial\theta}\frac{\langle Q\rangle}{V} \qquad \chi_{\mathrm{peak}}(L) = c_0 + c_1 L^p$$

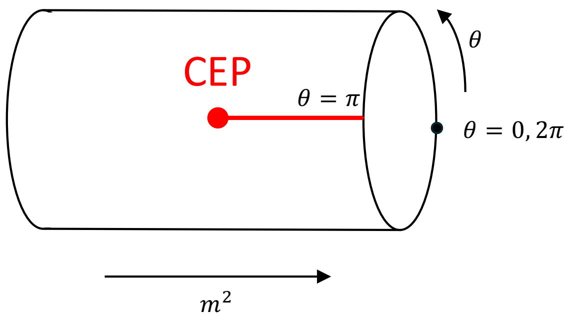


Fit results: p=2.00001(6), c_0=−0.0196(6), c_1=0.0009944(3)

⇒ 1st order phase transition (∵ $\chi_{\mathrm{peak}}(L) \propto V$)

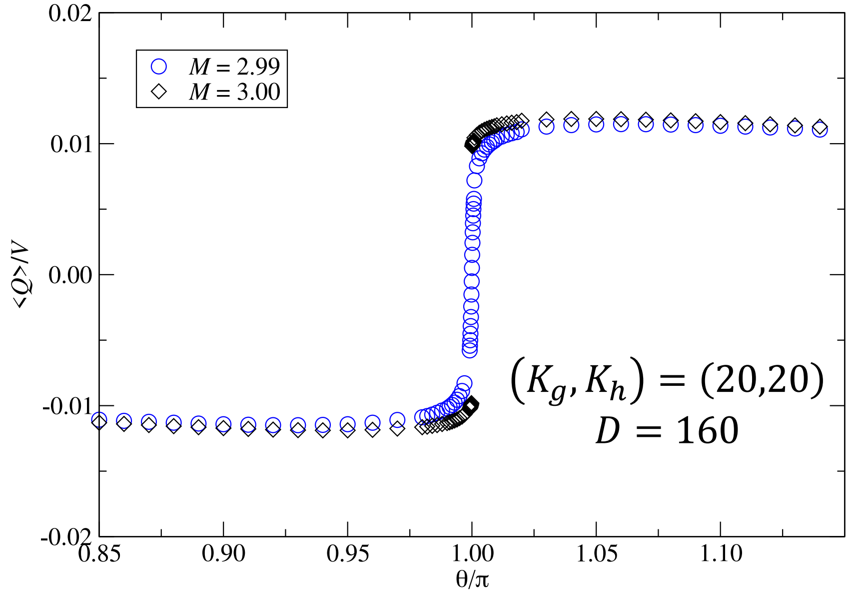
U(1) Gauge-Higgs Model w/ θ-term (1)

Determination of critical endpoint M_c at $(\beta, \epsilon, \lambda) = (3.0, 1.0, 0.5)$

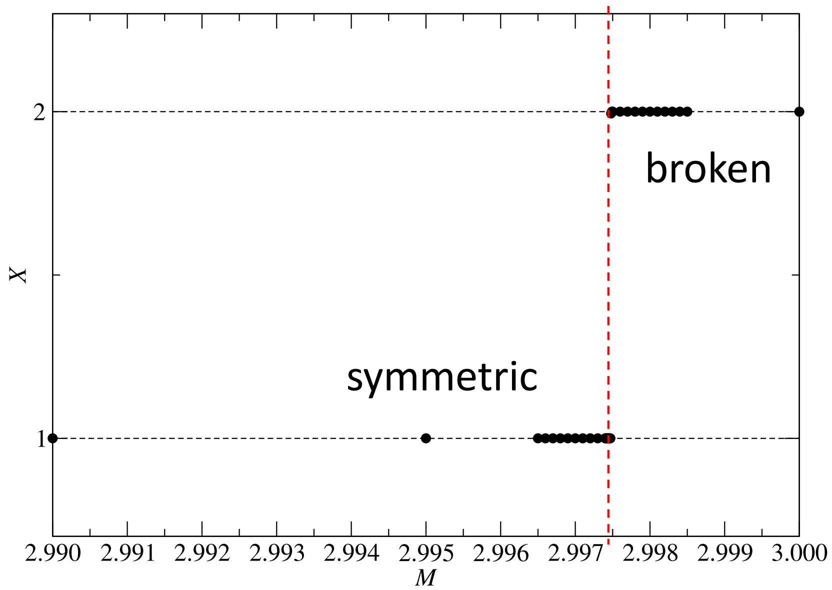


$M = m^2 + 4$

topological charge density



ground state degeneracy@θ=π



$\frac{\langle Q \rangle}{V}$: $2.99 \le M_c \le 3.00$

X: $2.99747 \le M_c \le 2.99748$

U(1) Gauge-Higgs Model w/ θ-term (2)

Identification of universality class

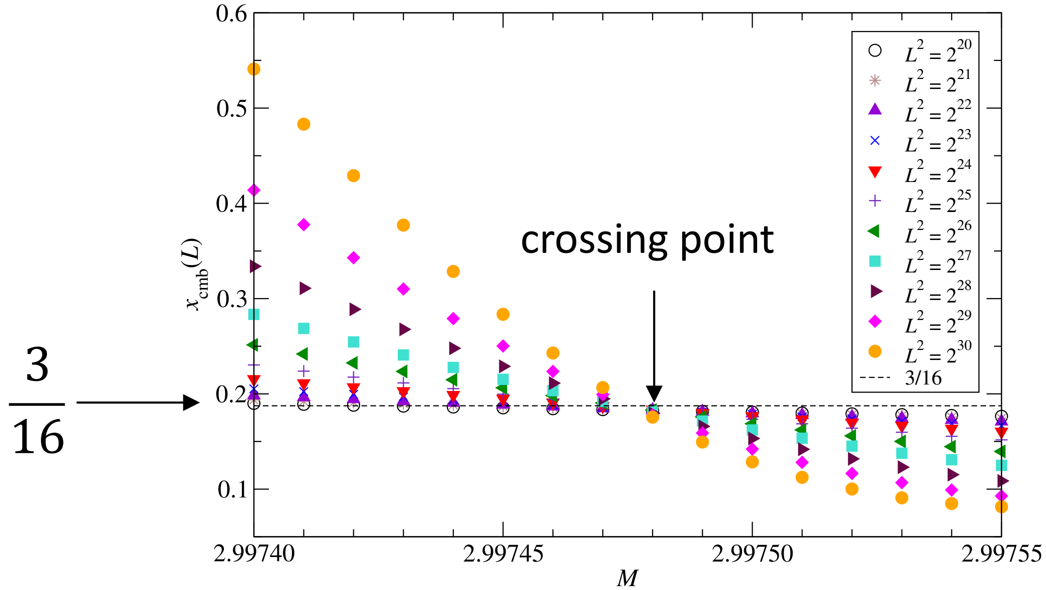
Use of level spectroscopy

Ueda-Oshikawa, PRB104(2021)165132,
PRB108(2023)024413

eigenvalues of transfer matrix: $\lambda_0 \geq \lambda_1 \geq \lambda_2 \geq \cdots$

scaling dimension: $x_n(L) = \frac{1}{2\pi} \ln \frac{\lambda_0(L)}{\lambda_n(L)}$

L dependence of $x_{\text{cmb}}(L) = x_1(L) + \frac{1}{16} x_2(L)$



Crossing point at $M_c = 2.99748$ agrees w/ $2.99747 \leq M_c \leq 2.99748$ by X

Summary

(1+1)d U(1) gauge-Higgs model w/ θ-term

- Gauge action is constructed w/ Lüscher's admissibility condition
- Suffers from the complex action problem and the topological freezing problem
- TRG is free from both problems
- Critical endpoint at $\theta = \pi$ is successfully determined
- Confirm the scaling dimension is consistent w/ the 2d Ising universality class

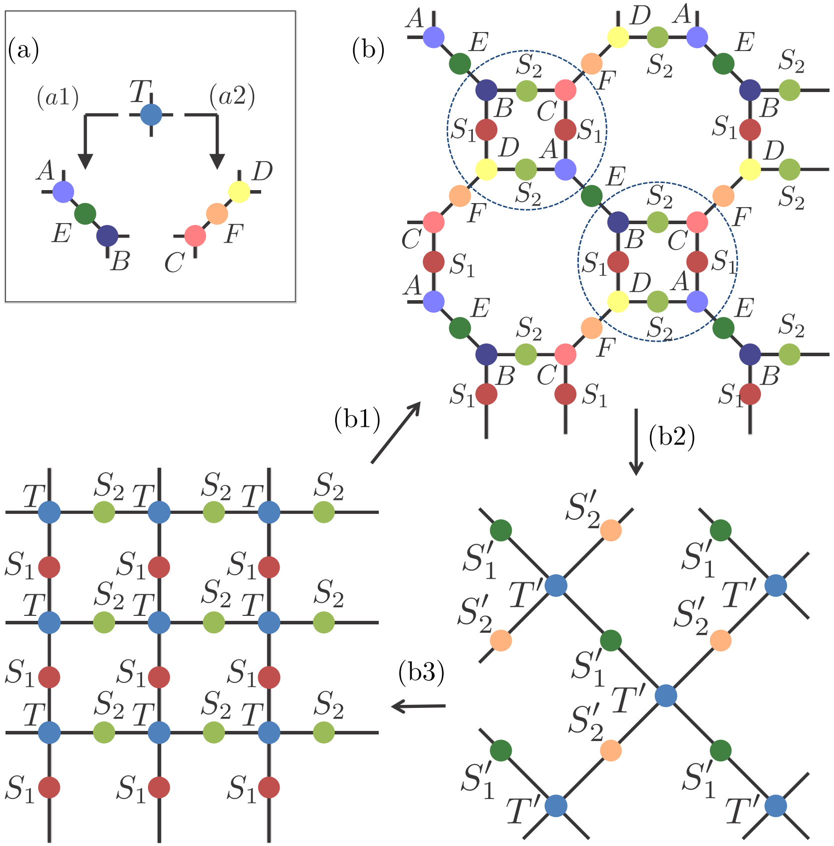
BACKUP

Bond-Weighted TRG (1)

Adachi-Okubo-Todo, PRB105(2022)L0604020

Levin–Nave TRG: $T \approx U\sigma V \Rightarrow (U\sqrt{\sigma})(\sqrt{\sigma}V)$

Bond–Weighted TRG (BTRG): $T \approx U\sigma V \Rightarrow \left(U\sigma^{\frac{1-k}{2}}\right)\sigma^{k}\left(\sigma^{\frac{1-k}{2}}V\right)$



$$A_{(x_0,y_0),i} = U_{1\,(x_0,y_0),i}\,\sigma_{1\,ii}^{(1-k)/2},$$

$$E_{i,j} = \delta_{ij}\,\sigma_{1\,ii}^{k},$$

$$B_{i,(x_1,y_1)} = \sigma_{1\,ii}^{(1-k)/2}\,V_{1\,i,(x_1,y_1)},$$

$$C_{(x_0,y_1),i} = U_{2\,(x_0,y_1),i}\,\sigma_{2\,ii}^{(1-k)/2},$$

$$F_{i,j} = \delta_{ij}\,\sigma_{2\,ii}^{k},$$

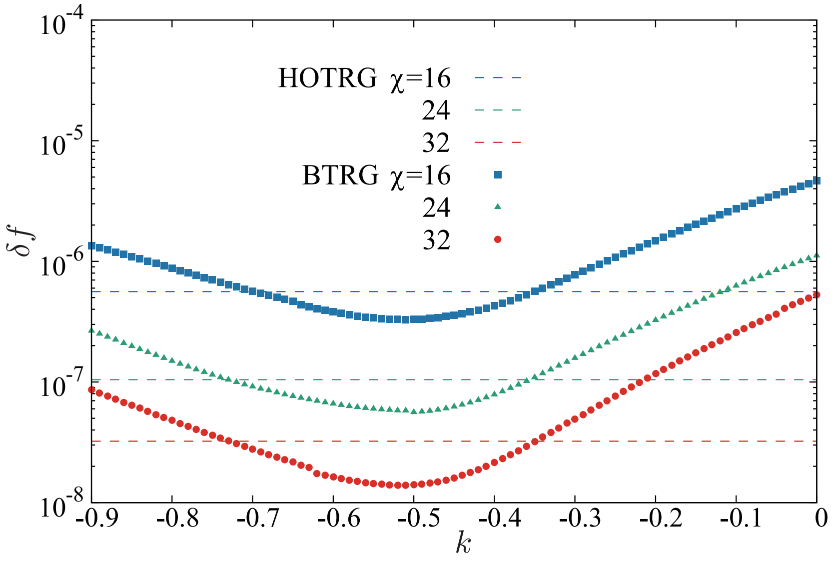
$$D_{i,(x_1,y_0)} = \sigma_{2\,ii}^{(1-k)/2}\,V_{2\,i,(x_1,y_0)}.$$

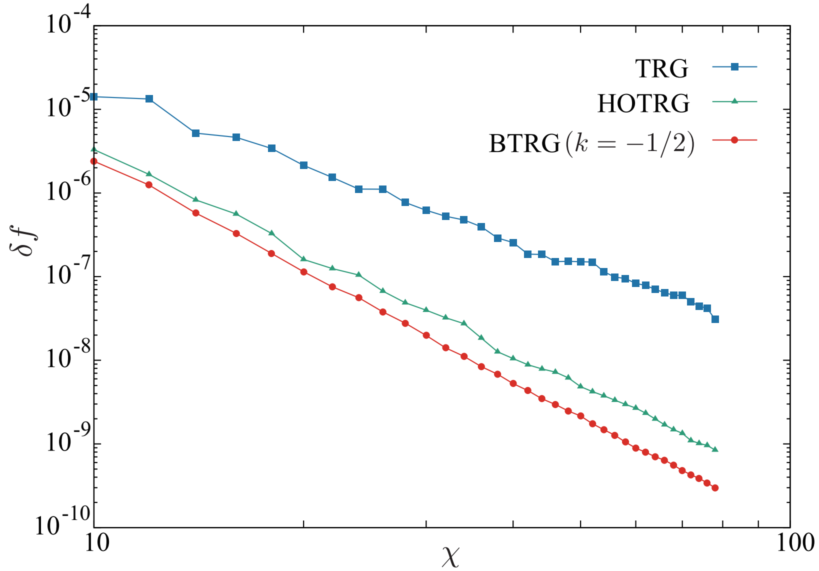
Bond-Weighted TRG (2)

Comparison of performance

2d Ising model at the critical point on $2^{16} \times 2^{16}$ lattice

$$\delta f = \frac{\|f_{\text{calc}} - f_{\text{exact}}\|}{\|f_{\text{exact}}\|}$$





BTRG outperforms HOTRG and conventional TRG

$k = -\frac{1}{2}$ seems optimal