A general method of the initial tensor construction for the TRG as the Steiner tree problem.

[K. N. arXiv:2307.14191]
[K. N., M.Schneider arXiv:2407.14226]

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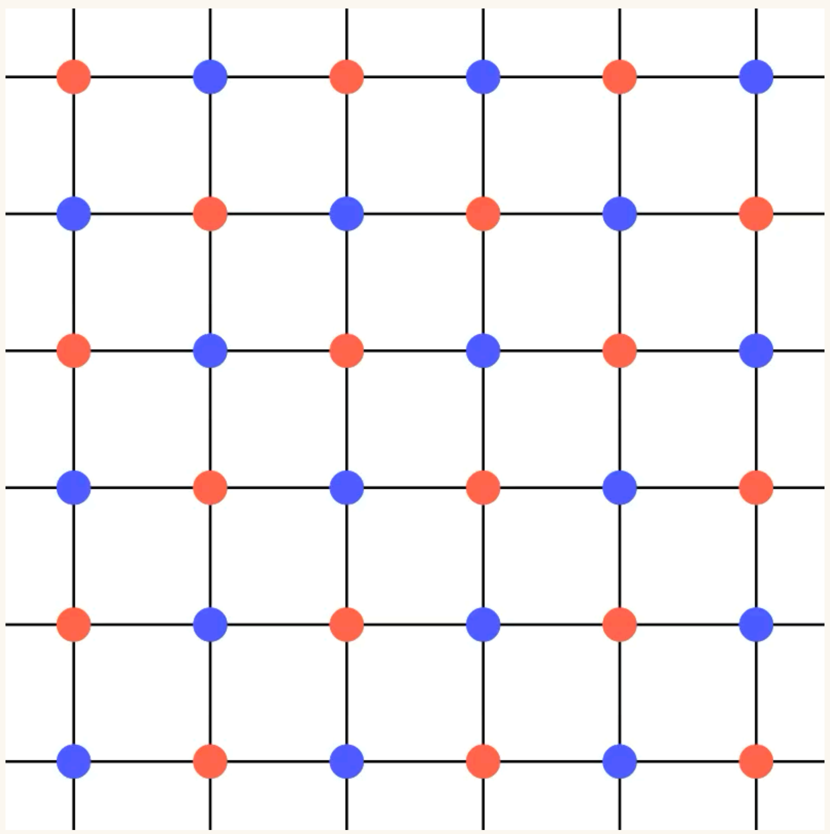
◉ Tensor renormalization group (TRG)

◇ What is the TRG?

→ TRG is approximated contraction of (locally connected) Tensor network.



[https://smorita.github.io/TN_animation/]

→ TRG require typical form of the tensor representation.

(Should be represented by graph)

$$Z = \mathrm{Tr} \sum_{i \in \mathrm{lattice}} A_{x_i y_i x'_i y'_i}$$

◇ What is the Tensor network?

→ Tensors are represented by each lattice points.

→ Indices are represented by line segments.

= Each index is only contained in two tensors.

Initial tensor network construction

Tensor network rep. of 2dim Ising

◇ How to find tensor network representation?

(e.g.): 2dim Ising model (Partition function)

Boltzmann factor

$$Z = \sum_{\sigma} \prod_{x,y} e^{\sigma_{x,y}\sigma_{x+1,y} + \sigma_{x,y}\sigma_{x,y+1}} = \sum_{\sigma} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}$$

→ Tensors $K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}$ do not construct tensor network.

(Index $\sigma_{x,y}$ is included in three tensors)

$$K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}}, K_{\sigma_{x-1,y}\sigma_{x,y}\sigma_{x-1,y+1}}, K_{\sigma_{x,y-1}\sigma_{x+1,y-1}\sigma_{x,y}}$$

↓

◇ Common method: (Taylor) expansion. and $\sigma^2 = 1$

Tensor network rep. of 2dim Ising

[H.H.Zhao et al. arXiv:1002.1405]

[Z.Y.Xie et al. arXiv:1201.1144] [Y.Liu et al. arXiv:1307.6543]

◇ Common method: (Taylor) expansion. and $\sigma^2 = 1$

$$e^{\beta\sigma_{x,y}\sigma_{x+1,y}} = \sum_{l_{x,y}=0}^{1} (\cosh(\beta))^{1-l_{x,y}}(\sigma_{x,y}\sigma_{x+1,y}\sinh(\beta))^{l_{x,y}} = \sum_{l_{x,y}=0}^{1} W_{\sigma_{x,y}l_{x,y}} W_{\sigma_{x+1,y}l_{x,y}}$$

x-direction: $\sigma \to l$

y-direction: $\sigma \to m$

$$W = \begin{pmatrix} \sqrt{\cosh(\beta)}, & \sqrt{\sinh(\beta)} \\ \sqrt{\cosh(\beta)}, & -\sqrt{\sinh(\beta)} \end{pmatrix}$$

$$Z = \sum_{\sigma} \prod_{x,y} e^{\sigma_{x,y}\sigma_{x+1,y}+\sigma_{x,y}\sigma_{x,y+1}}$$

Taking summation of σ first.

$$= \sum_{\sigma,l,m} \prod_{x,y} W_{\sigma_{x,y},l_{x,y}} W_{\sigma_{x,y},l_{x-1,y}} W_{\sigma_{x,y},m_{x,y}} W_{\sigma_{x,y},m_{x,y-1}} = \sum_{l,m} \prod_{x,y} K^{(\mathrm{exp})}_{l_{x,y}l_{x-1,y}m_{x,y}m_{x,y-1}}$$

→ Tensors $K^{(\mathrm{exp})}_{l_{x,y}l_{x-1,y}m_{x,y}m_{x,y-1}}$ construct tensor network.

(Index $l_{x,y}$ and $m_{x,y}$ are included in two tensors)

$$K^{(\mathrm{exp})}_{l_{x,y}l_{x-1,y}m_{x,y}m_{x,y-1}}, K^{(\mathrm{exp})}_{l_{x+1,y}l_{x,y}m_{x+1,y}m_{x+1,y-1}}$$

◉ Tensor network rep. by delta matrix

[K. Nakayama, M.Schneider arXiv:2407.14226]

◇ Key point: Expansion produces tensor network rep. using property, $\sigma^2 = 1$.

→ Problem: More complicated Boltzmann factor produces complicated form (difficult to find the tensor network).

◇ Our proposal: Index shift by delta matrix.

(e.g.): 2dim Ising model (periodic b.c.)

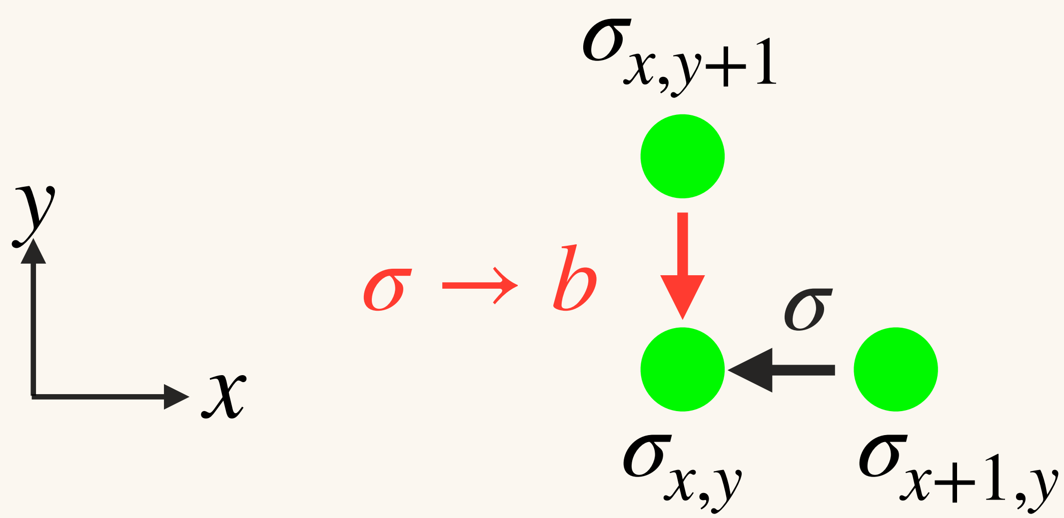
$$Z = \sum_{\sigma} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} = \sum_{\sigma,b} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \delta_{b_{x,y+1}\sigma_{x,y+1}} = \sum_{\sigma,b} \prod_{x,y} K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \delta_{b_{x,y}\sigma_{x,y}}$$

$$= \sum_{\sigma,b} \prod_{x,y} K^{(\mathrm{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}}$$

◆ index shift $(y+1 \to y)$ by δ

→ Tensor network constructed by $K^{(\mathrm{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}} \equiv K_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y+1}} \times \delta_{b_{x,y}\sigma_{x,y}}$

◉ Schematic picture of the construction



$$K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} \to K^{(\text{delta})}_{\sigma_{x,y}\sigma_{x+1,y}b_{x,y}b_{x,y+1}}$$

(1): Dots ↔ original indices, $\{\sigma_{x,y}, \sigma_{x+1,y}, \sigma_{x,y+1}\}$

(2): Draw arrows to connect all dots.

(3): Arrow (except σ) ↔ Index shift by δ.

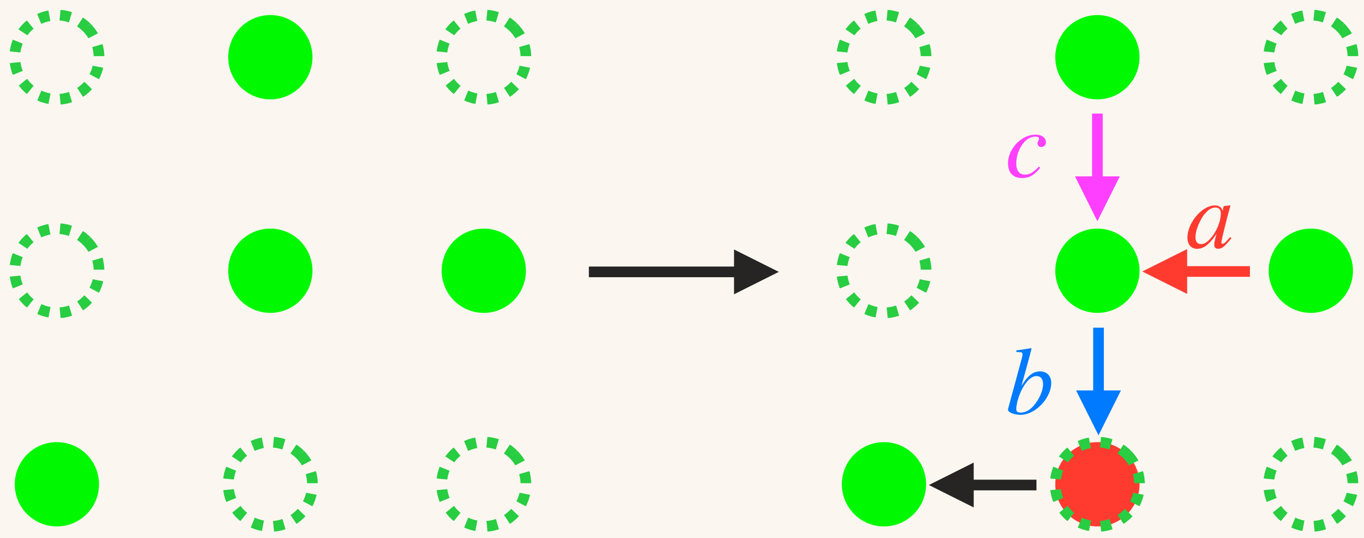
↓

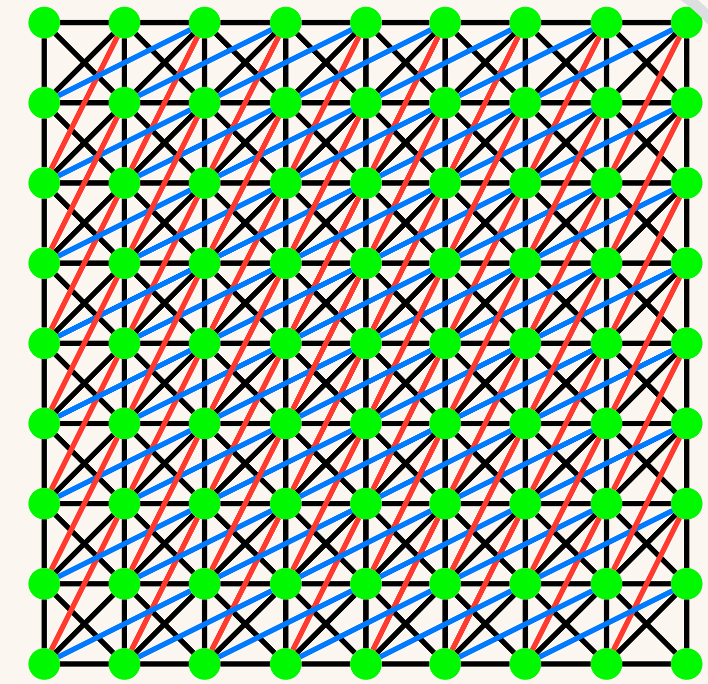
Arrows ↔ new indices, $\{\sigma, a\}$.

→ This method don't require any properties (equations).

(e.g. $\sigma^2 = 1$, $K_{\sigma_{x,y}\sigma_{x+1,y}\sigma_{x,y+1}} = e^{\sigma_{x,y}\sigma_{x+1,y}+\sigma_{x,y}\sigma_{x,y+1}}$)

◉ Generalization: J1-J2+α





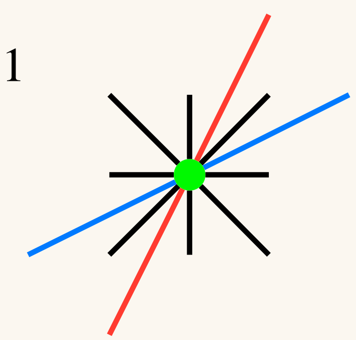
$$K_{\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}} \rightarrow K''_{[\sigma a]_{x,y}[\sigma a]_{x+1,y}[bc]_{x,y}[bc]_{x,y+1}}$$

$$e^{h(\sigma_{x,y}+\sigma_{x+1,y+1}+\sigma_{x+2,y+1}+\sigma_{x+1,y+2})}$$

$$e^{J_1(\sigma_{x+1,y+1}\sigma_{x+2,y+1}+\sigma_{x+1,y+1}\sigma_{x+1,y+2})}\, e^{J_2(\sigma_{x,y}\sigma_{x+1,y+1}+\sigma_{x+2,y+1}\sigma_{x+1,y+2})}\, e^{g_1\sigma_{x,y}\sigma_{x+2,y+1}+g_2\sigma_{x,y}\sigma_{x+1,y+2}}$$

$$e^{t_1\sigma_{x,y}\sigma_{x+2,y+1}\sigma_{x+1,y+2}+t_2\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}+t_3\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+1,y+2}+t_4\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$

$$e^{e_1\sigma_{x,y}\sigma_{x+1,y+1}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$



→ Added red dot is Steiner point.

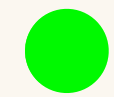
→ In general: Rectilinear Steiner tree problem (generalization of the traveling salesman problem).

→ K'' has only $4 \times 4 \times 4 \times 4$ indices.

◉ J1-J2+α: Frustrated system with 12 parameters.

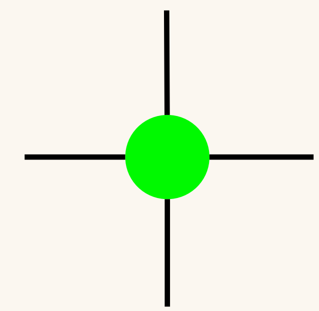
1-spin

$$e^{h(\sigma_{x,y}+\sigma_{x+1,y+1}+\sigma_{x+2,y+1}+\sigma_{x+1,y+2})}$$



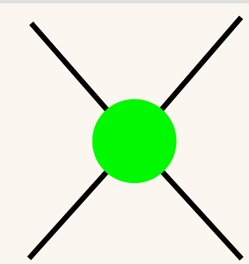
2-spin (J1)

$$e^{J_1^{(x)}\sigma_{x+1,y+1}\sigma_{x+2,y+1}+J_1^{(y)}\sigma_{x+1,y+1}\sigma_{x+1,y+2}}$$



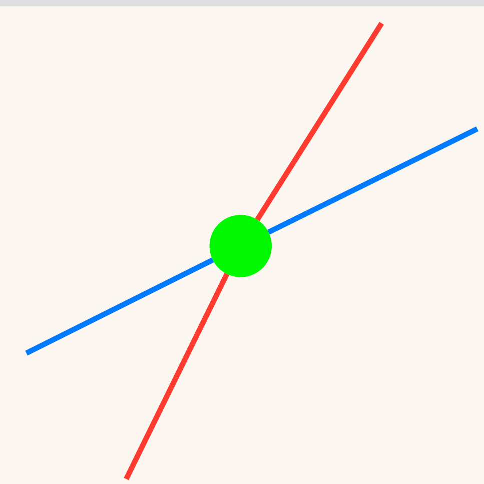
2-spin (J2)

$$e^{J_2^{(a)}\sigma_{x,y}\sigma_{x+1,y+1}+J_2^{(b)}\sigma_{x+2,y+1}\sigma_{x+1,y+2}}$$



2-spin (g1,g2)

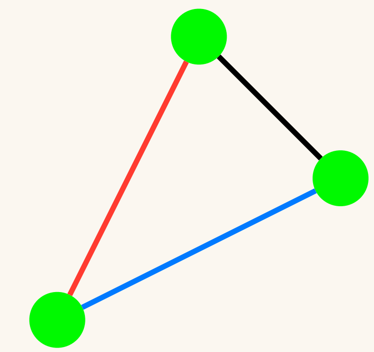
$$e^{g_1\sigma_{x,y}\sigma_{x+2,y+1}+g_2\sigma_{x,y}\sigma_{x+1,y+2}}$$



◉ J1-J2+α: Frustrated system with 12 parameters.

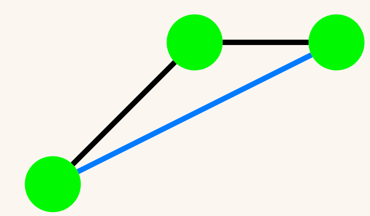
3-spin t1

$$e^{t_1 \sigma_{x,y} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$



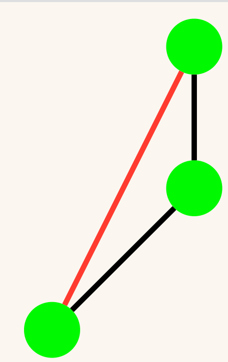
3-spin t2

$$e^{t_2 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+2,y+1}}$$



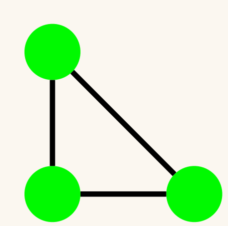
3-spin t3

$$e^{t_3 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+1,y+2}}$$



3-spin t4

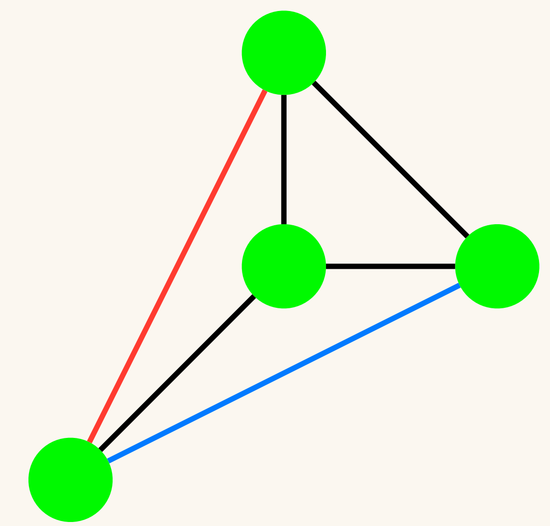
$$e^{t_4 \sigma_{x+1,y+1} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$

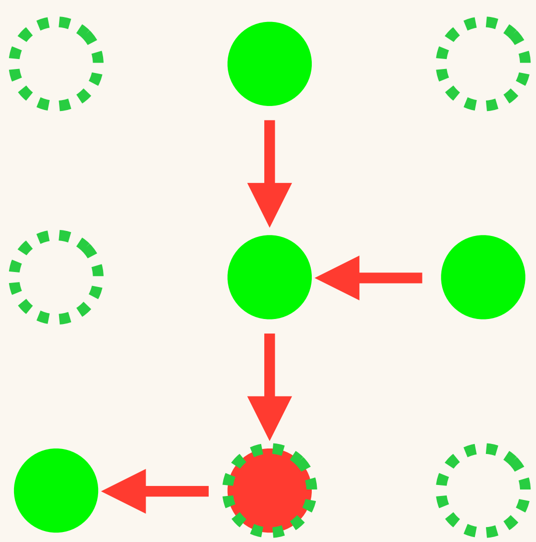


◉ J1-J2+α: Frustrated system with 12 parameters.

4-spin

$$e^{e_1 \sigma_{x,y} \sigma_{x+1,y+1} \sigma_{x+2,y+1} \sigma_{x+1,y+2}}$$





$$= K''_{[\sigma a]_{x,y}[\sigma a]_{x+1,y}[bc]_{x,y}[bc]_{x,y+1}}$$

◇ Num. of kind of interaction (parameters):

$$\left(\sum_{k=2}^{4} {}_4C_k \right) + 1 = 2^4 - 4 = 12$$

→ K'' has only $4 \times 4 \times 4 \times 4$ indices.

◉ Note: Traveling salesman problem.

◇ Traveling salesman problem (famous NP-hard):
Find shortest line segments (roads) between dots (towns) to connect every dots.

↓

◇ Steiner tree problem:
Traveling salesman + we can freely add dots (town)

↓

◇ Rectilinear Steiner tree problem: (NP-hard (complete))
Steiner tree on the lattice.
…dots on lattice points, line segments only on the links.

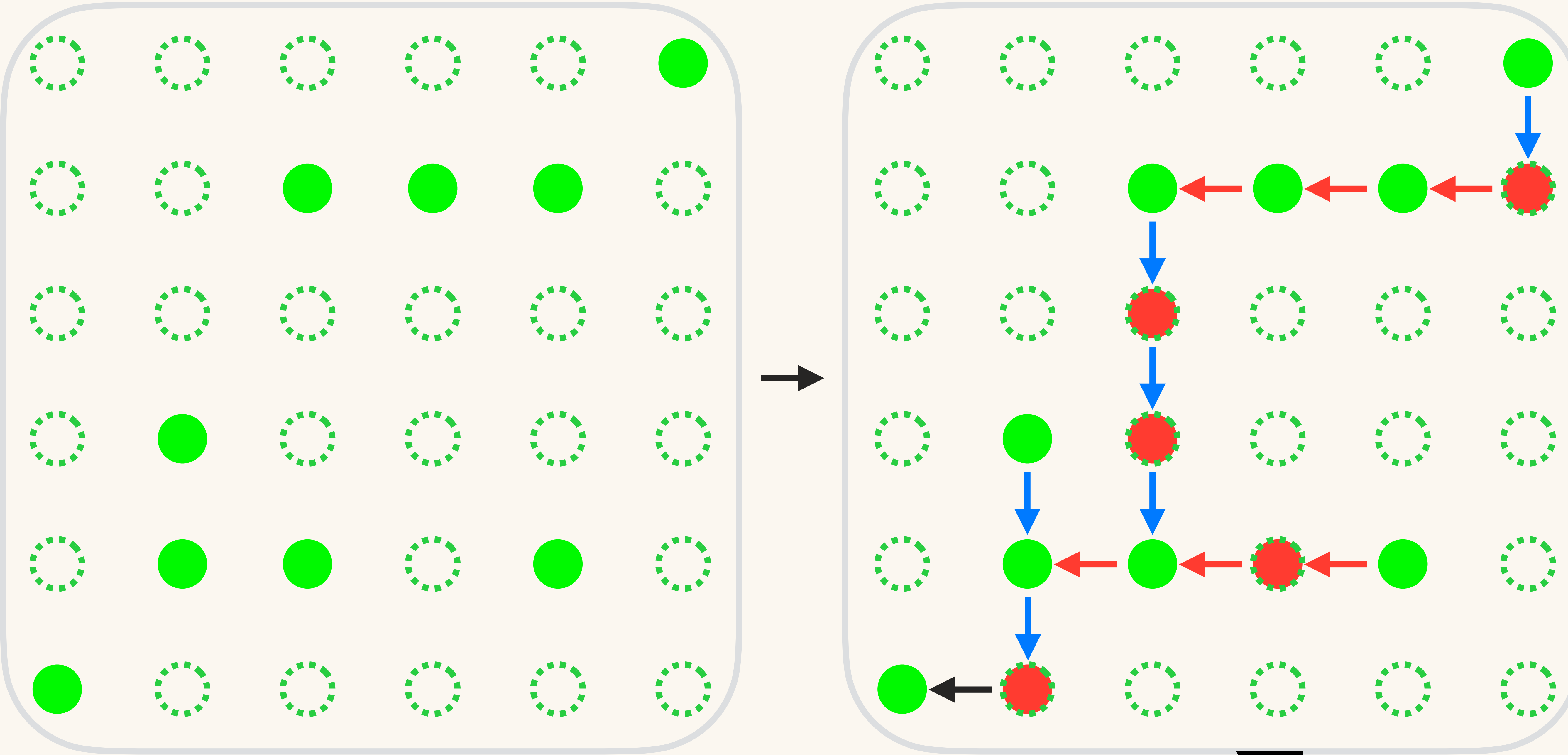
→ Index size of tensor = $D_{\mathrm{ini}}^{(\mathrm{Road\ length})}$

→ Long range interaction becomes harder.

→ Mild long ranges (e.g. plaquette, clover, etc.) are not hard.

◉ Note: (Rectilinear) Steiner tree problem

◇ Rectilinear Steiner tree problem: Find shortest line segment on the lattice between dots with additional dots.



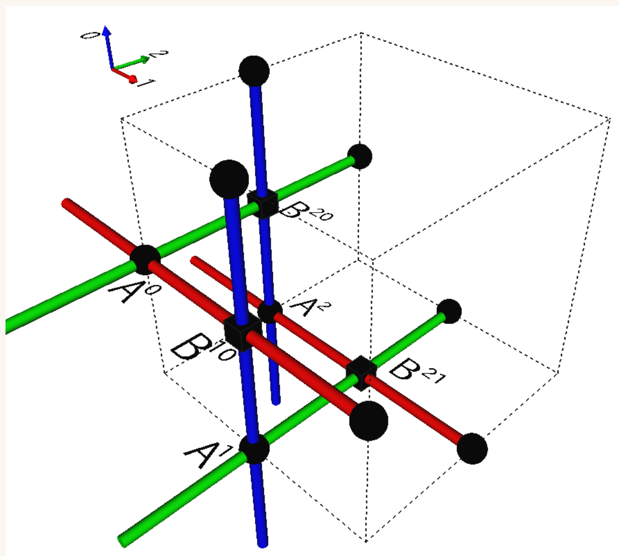
→ Num. of interaction (parameters) order $\sim O(\sum {}_9C_k = 2^9)$

→K'' has $2^6 \times 2^6 \times 2^7 \times 2^7$ indices.

◉ Tensor network rep. of 3dim Z2 gauge theory

[Y.Liu et al. arXiv:1307.6543] [Y.Kuramashi, Y.Yoshimura, arXiv:1808.08025]

◇ Common method: (Taylor) expansion. and $\sigma^2 = 1$



$$Z = 2^{-3V} \sum_{\sigma} \prod_{n,\mu>\nu} e^{-\beta \sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}}$$

$$e^{-\beta \sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}} = \cosh\beta \sum_{p=0}^{1} (\tanh\beta)^p (\sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu})^p$$

$$A_{pqrs} = \mathrm{mod}(1+p+q+r+s,2)$$

$$B_{pqrs} = (\tanh\beta)^{(p+q+r+s)/4}\delta_{pq}\delta_{pr}\delta_{rs}$$

↓

$$Z = \sum_{g,h,i,j,k,l} \prod_{n} T^{(\mathrm{exp})}_{[gh]_{x,y,z}[gh]_{x+1,y,z}[ij]_{x,y,z}[ij]_{x,y+1,z}[kl]_{x,y,z}[kl]_{x,y,z+1}}$$

$$T^{(\mathrm{exp})}_{[xX][x'X'][yY][y'Y'][zZ][z'Z']} = (\cosh\beta)^3 \sum_{a,b,c,d,e,f} A_{cyZe}A_{fzxb}A_{dYXa}B_{bx'y'c}B_{aX'Z'e}B_{fz'Y'd}$$

◇ In general: Any kind of expansion produces tensor network.

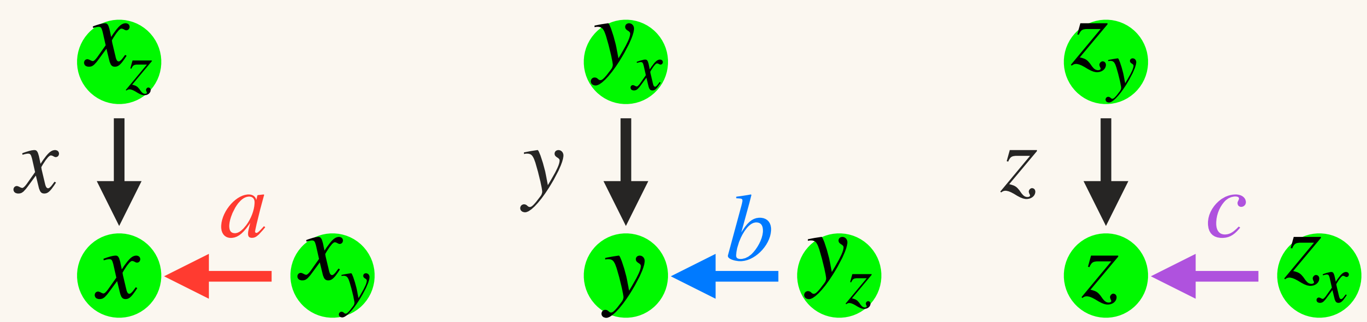
(Character for gauge theory, Orthogonal function, Taylor…)

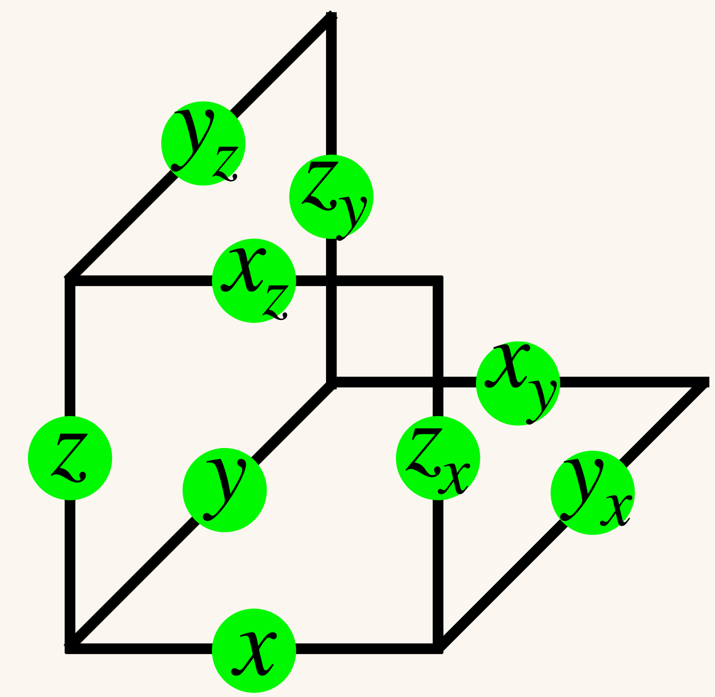
◉ Tensor network rep. of 3dim Z2 gauge theory

◇ Our proposal: Index shift by delta matrix.

$$Z = 2^{-3V}\sum_{\sigma}\prod_{n,\mu>\nu} e^{-\beta\sigma_{n,\mu}\sigma_{n+\hat{\mu},\nu}\sigma_{n+\hat{\nu},\mu}\sigma_{n,\nu}} = \sum_{\sigma}\prod_{n,\mu>\nu} e^{-\beta(xx_{\hat{y}}yy_{\hat{x}}+xx_{\hat{z}}zz_{\hat{x}}+yy_{\hat{z}}zz_{\hat{y}})/8}$$

→ x, y, z are independent with each other.



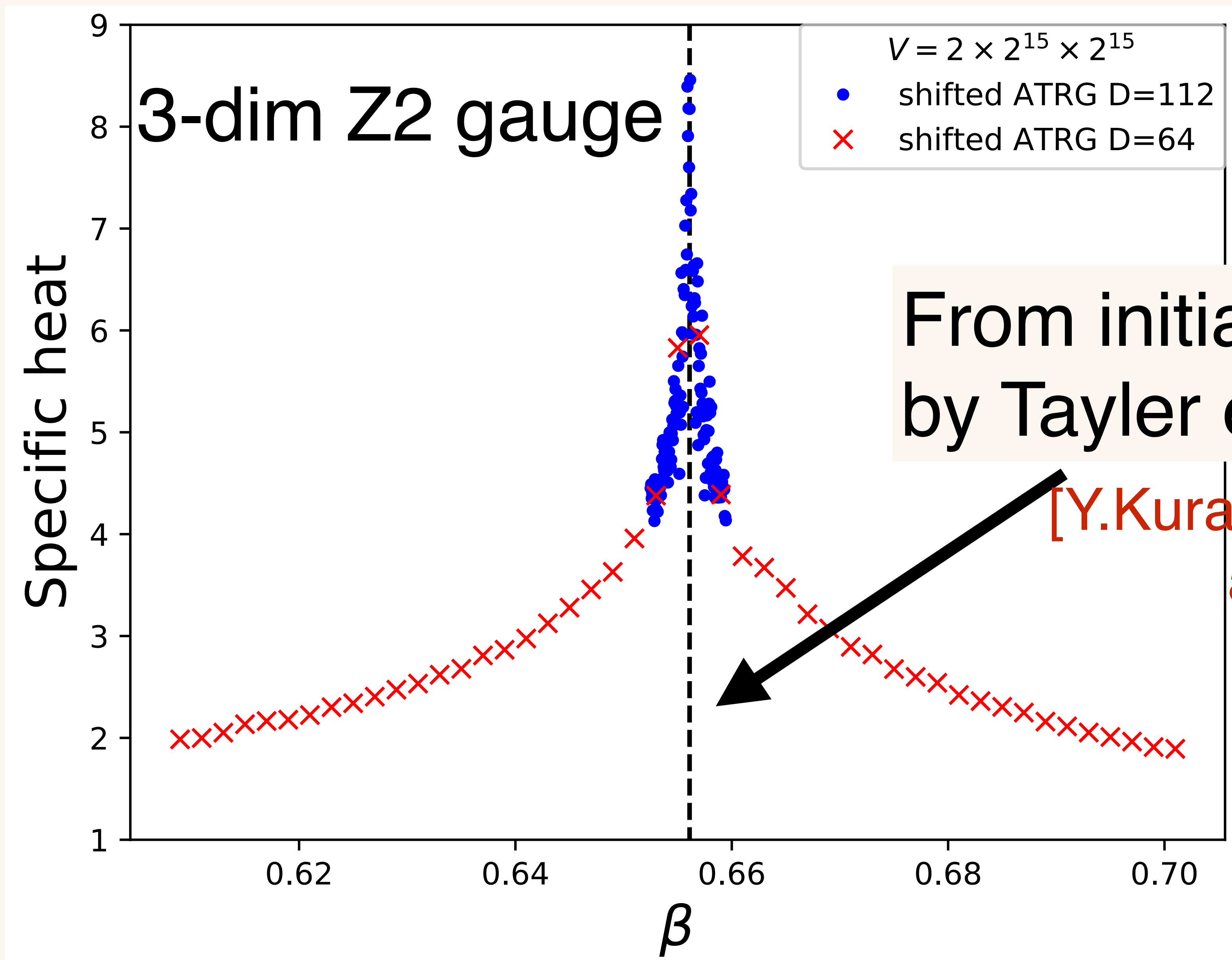


$$K_{xx_yx_zyy_zy_xzz_xz_y} \rightarrow K''_{[xb][xb]_z[yc][yc]_x[za][za]_y}$$

→ K'' has only $4 \times 4 \times 4 \times 4 \times 4 \times 4$ indices.

◉ Critical tempareture of 3dim Z2 gauge theory

◇ Numerical calc. by (modified) ATRG + impurity method



[Y.Kuramashi et al. arXiv:1808.08025]

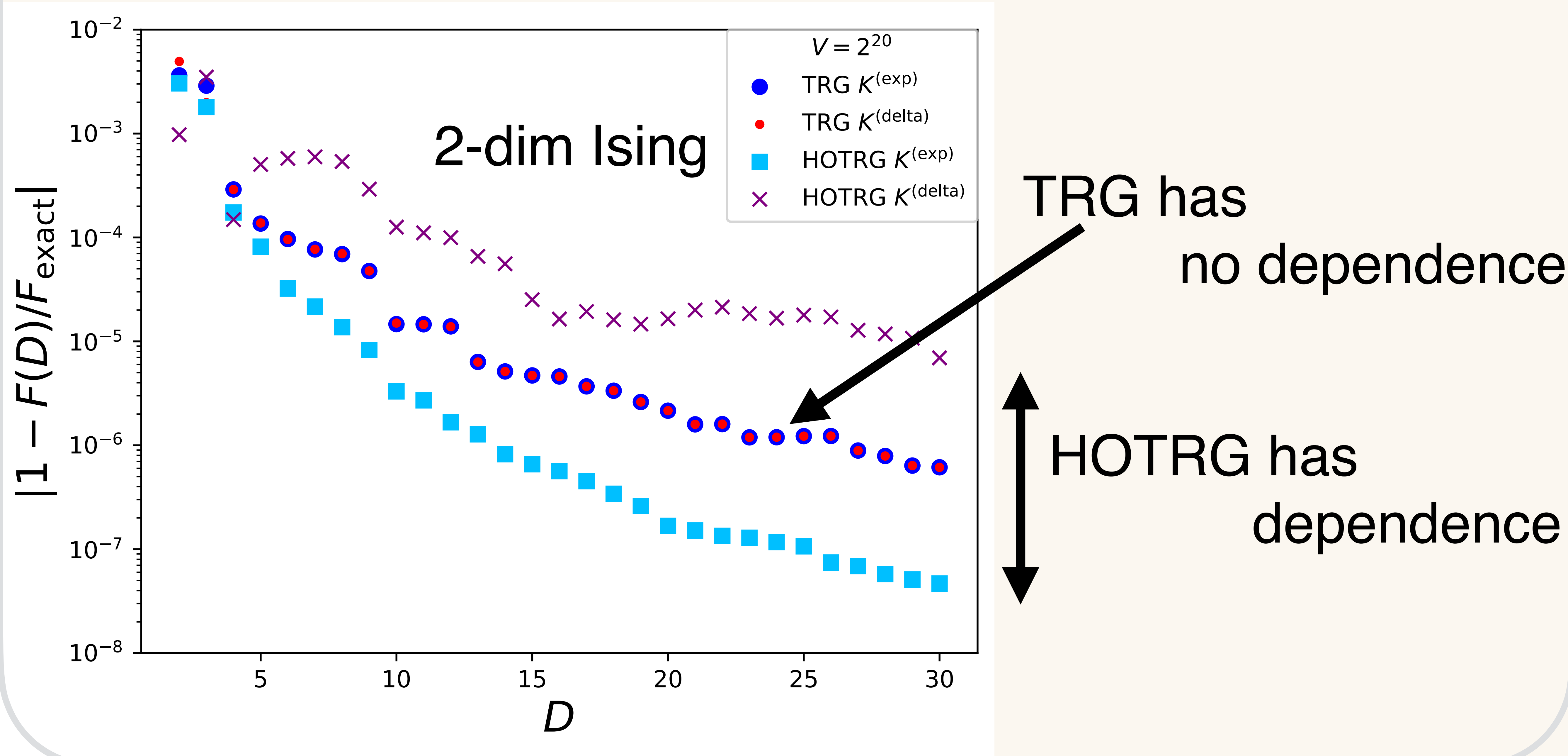
→ Our method produces correct result for critical temperature.

Initial tensor network dependence

Initial tensor dependence

◇ Now we can construct initial tensor by simple method.

→ We did not say our method produces best precision.

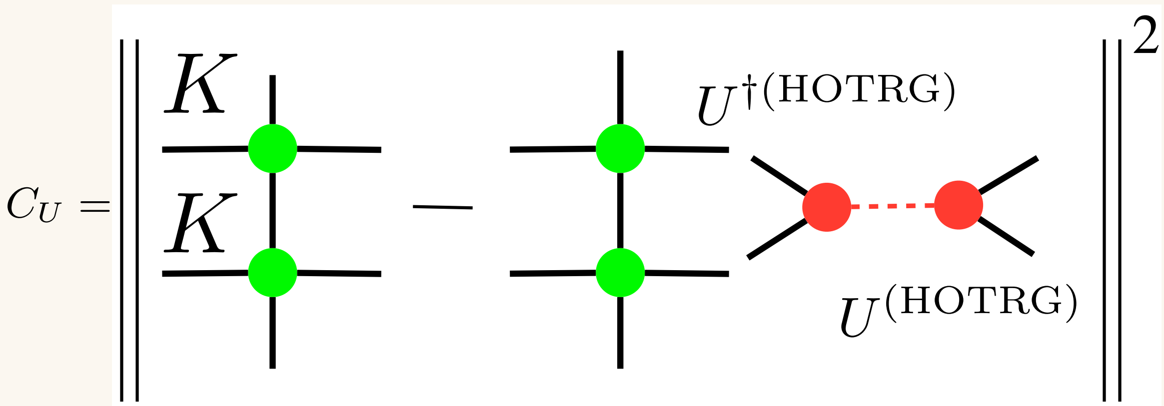


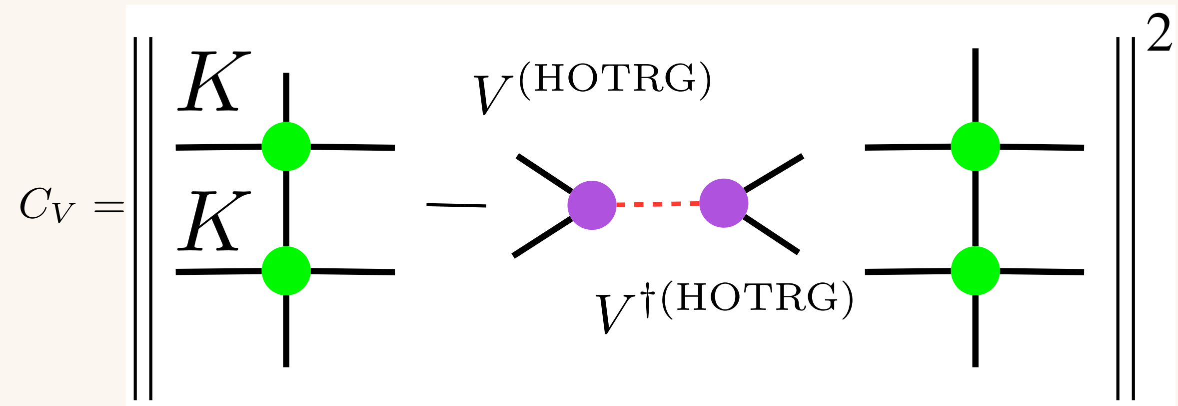
→ But we can eliminate this initial tensor dependence.

Boundary TRG

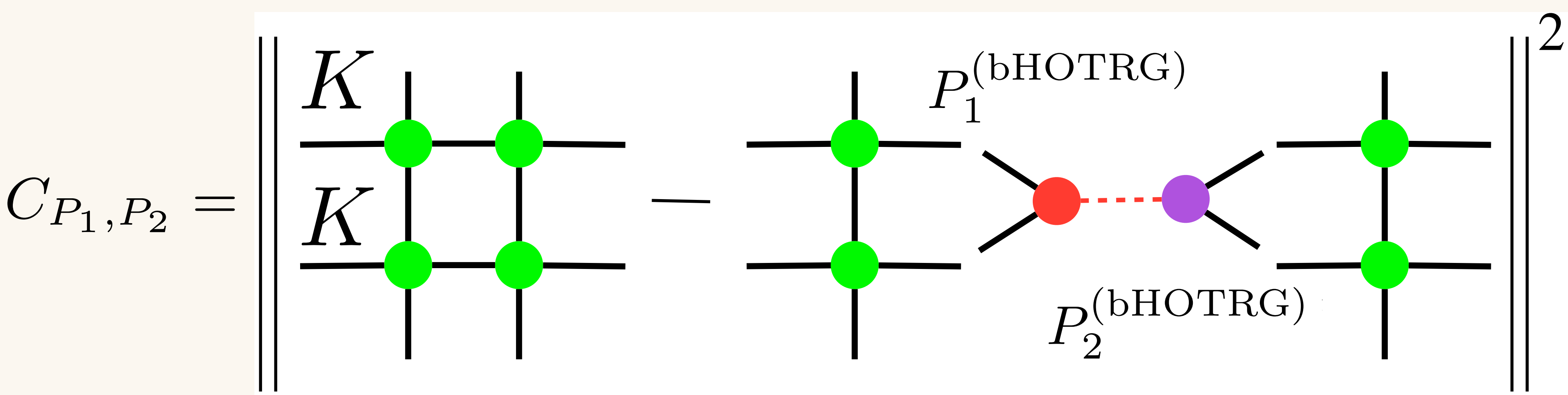
[S. Iino et al. arXiv:1905.02351]
[K. N., M.Schneider arXiv:2407.14226]

◇ Original HOTRG: choose U or V by minimizing the norm.



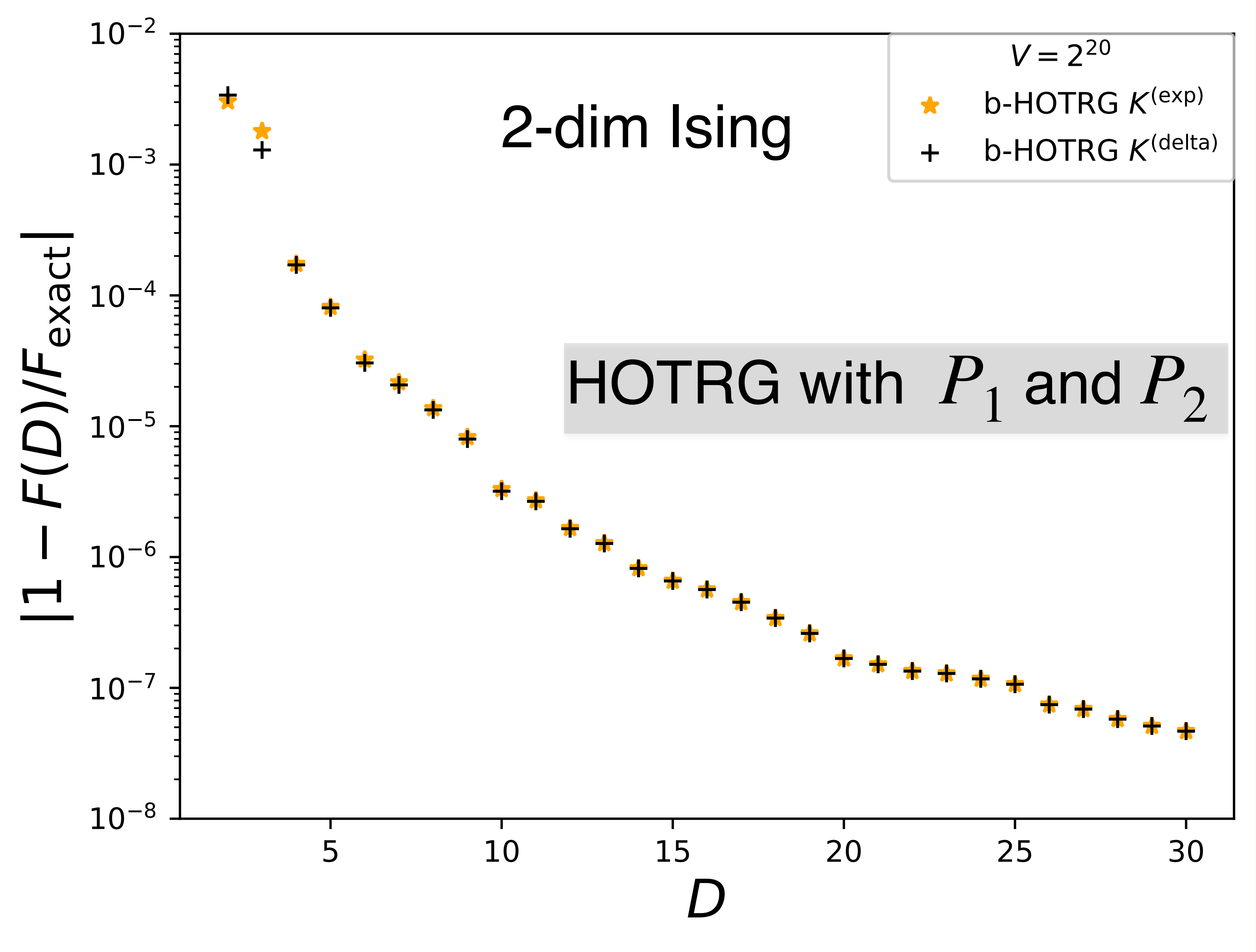


◇ HOTRG with boundary TRG:
construct P_1, P_2 by U and V.



→ Generalization from U or V to P_1 and P_2 can be done for any other TRG methods.

◉ Boundary HOTRG



→ HOTRG with P_1 and P_2 is initial tensor independent.

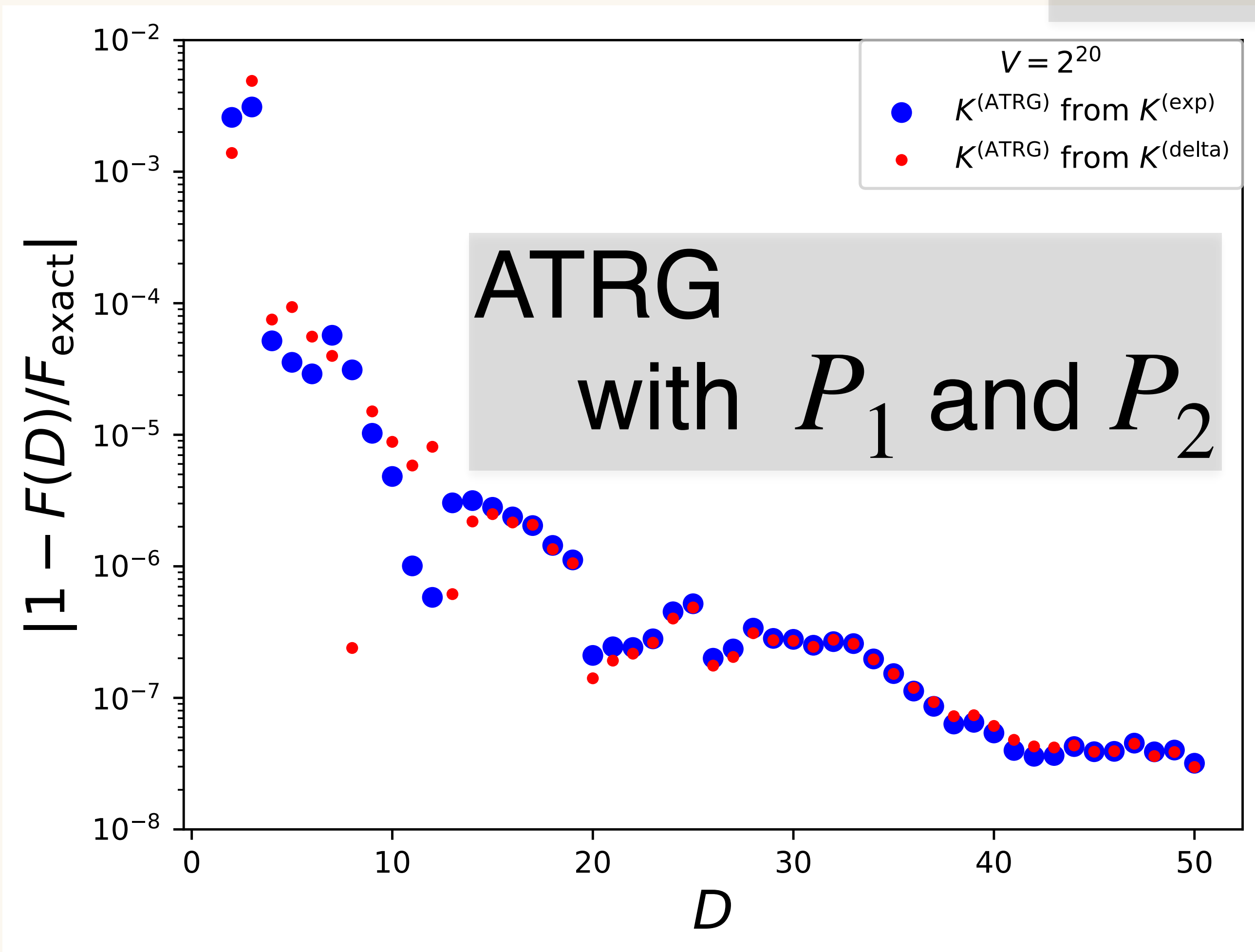
◉ Boundary TRG for ATRG and MDTRG

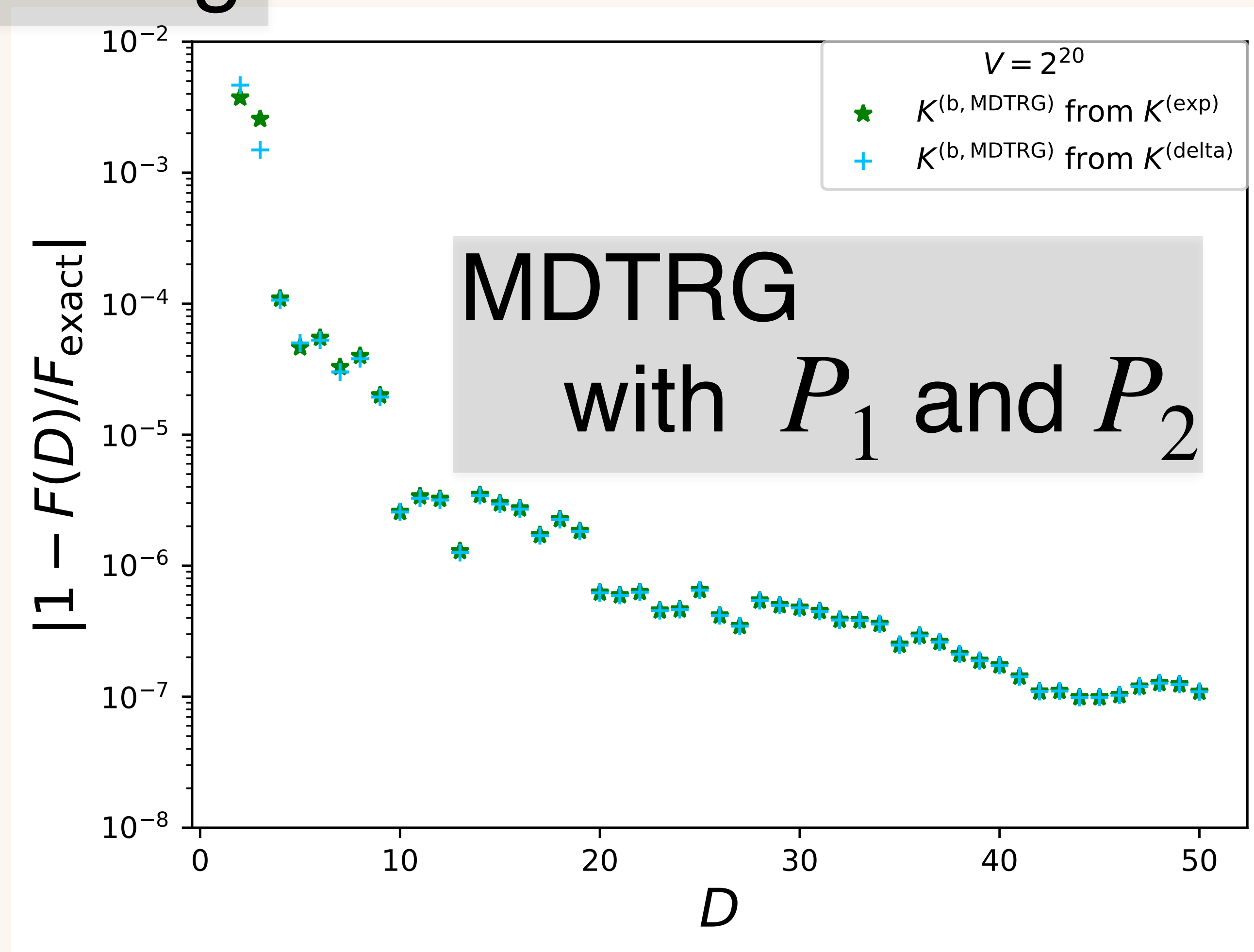
[D. Adachi et al. arXiv:1906.02007]
[D. Kadoh, K.N. arXiv:1912.02414] [K.N. arXiv:2307.14191]

- ◇ Many method (ATRG, TriadTRG, MDTRG) can use isometry (U or V) or P_1 and P_2.

[K. N., M.Schneider arXiv:2407.14226]

2-dim Ising





→ TRG methods with P_1 and P_2 are initial tensor independent.

→ Our construction can also produce compatible result.

Summary

[main]

- We propose a general method to construct the initial tensor network from Boltzmann factor representation.
- We can eliminate initial tensor dependence by using boundary TRG method for isometry.

[details]

- This relates to the rectilinear Steiner tree problem (generalized traveling salesman problem)
- We test our construction in the 2-dim Ising model and 3-dim Z2 gauge theory, and reproduce results.

→ Our method could be a simple, good choice for first study.

Backup

◉ Tensor renormalization group (TRG)

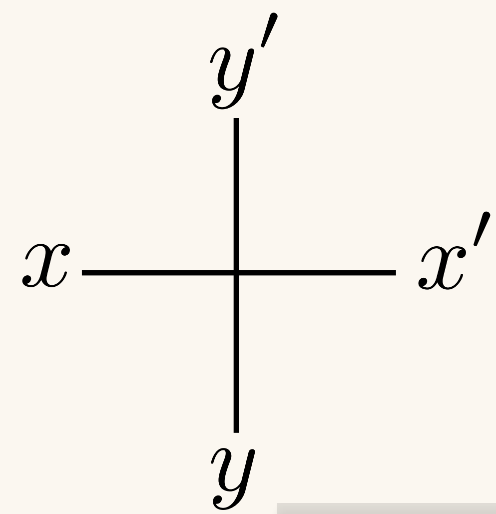
[M. Levin, C. P. Nave. arXiv:cond-mat/0611687]

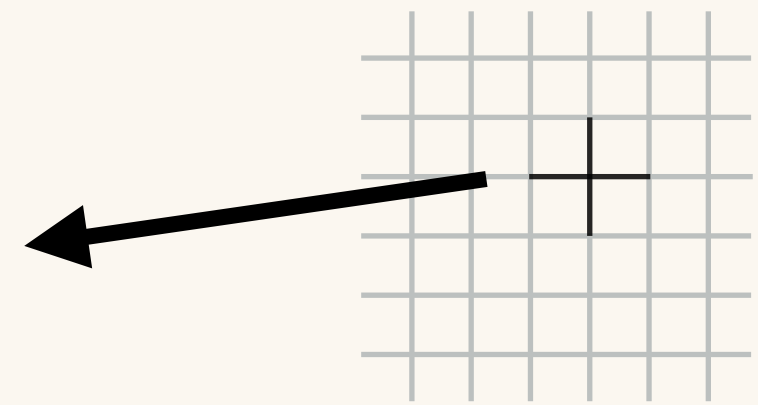
◇ TRG calculate the physical quantity as trace of tensors.

$$Z = \mathrm{Tr} \prod_{i \in \mathrm{lattice}} A_{x_i y_i x'_i y'_i}$$

$$\int D\overline{\psi} D\psi \left[-\sum_x \mathcal{L}(\overline{\psi}_x, \psi_x) \right]$$

$$A_{xyx'y'} =$$





○Sign problem

○Another representation

×High cost ($\dim \geq 3$)

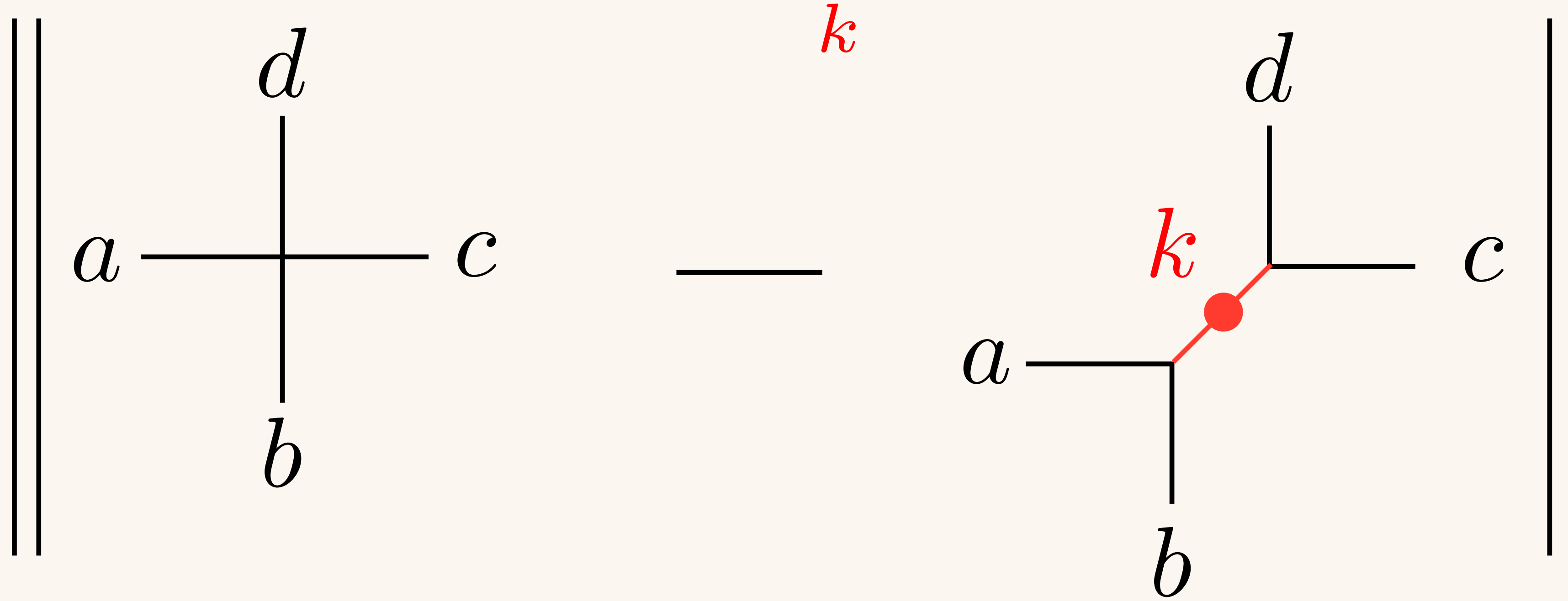
△Systematic error

How can we take whole contraction approximately?

→ Singular value decomposition (Frobenius norm)

◉ SVD and corresponding cost function

$$T_{abcd} = \sum_{k}^{D} A_{ab}{}^{k} \lambda^{k} B_{cd}{}^{k}$$

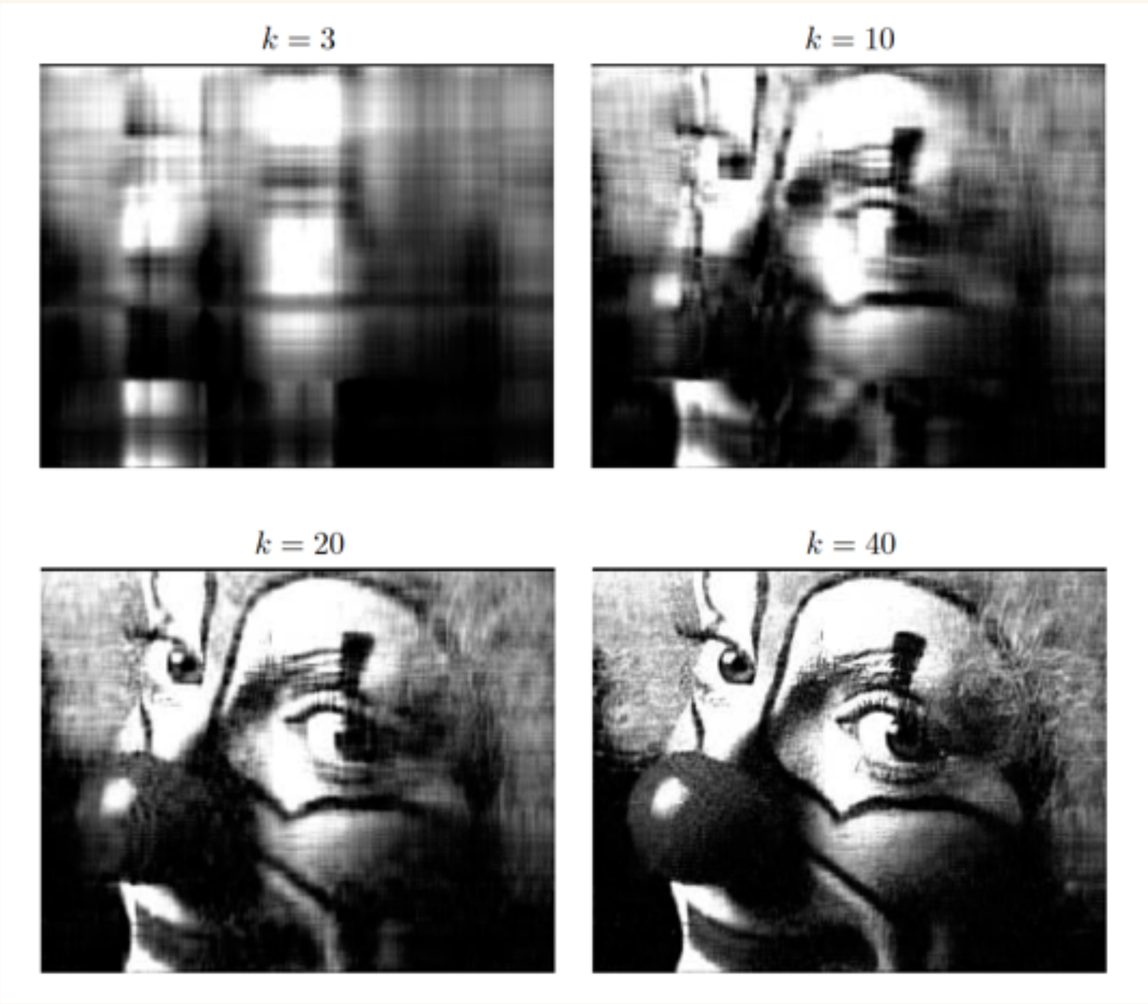


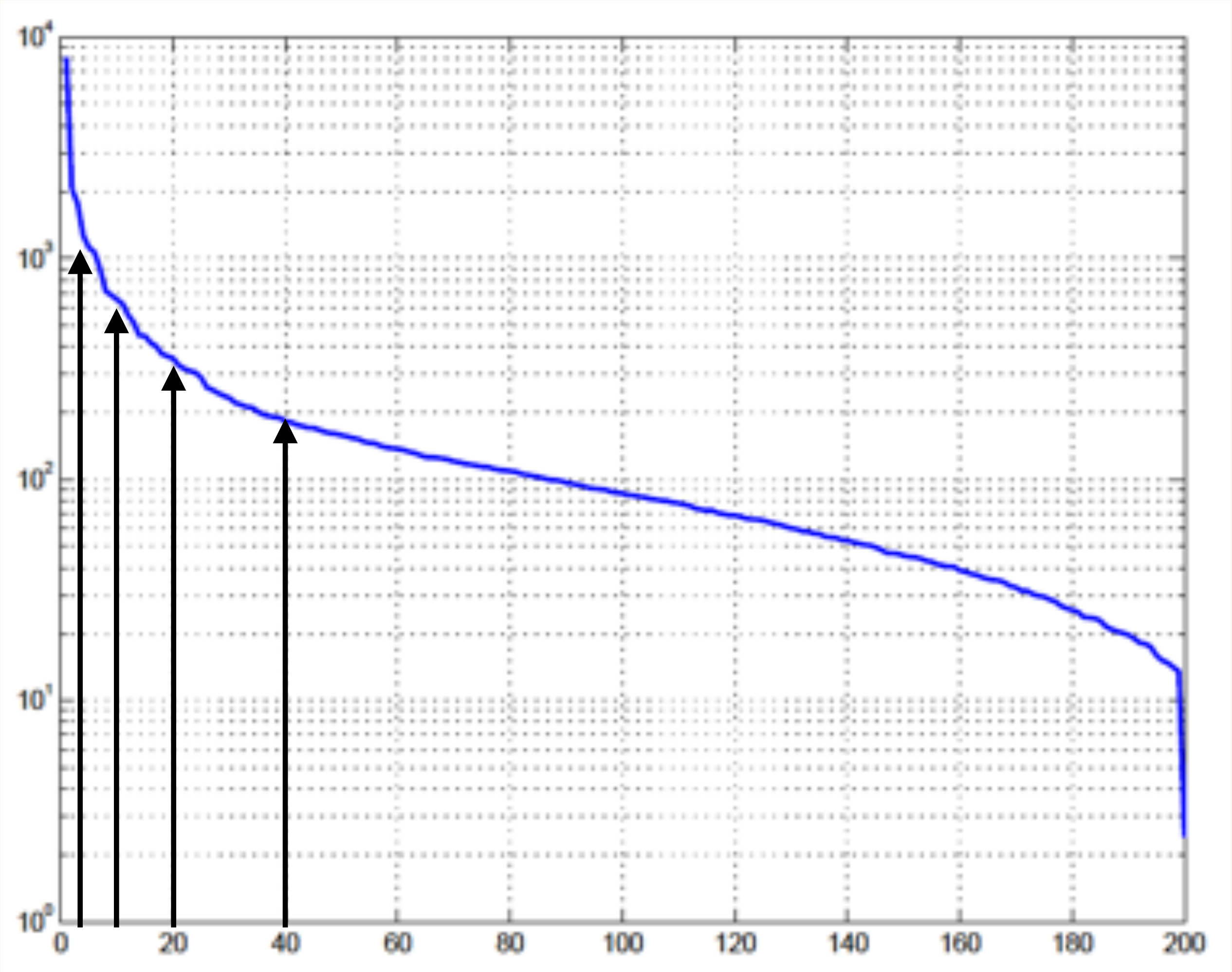
◇ truncation of the SVD minimize the cost function

→ k is truncated

$$\dim(k) = \dim(a)\dim(b) \to D$$

◉ SVD for a coarse graining (e.g. Image)



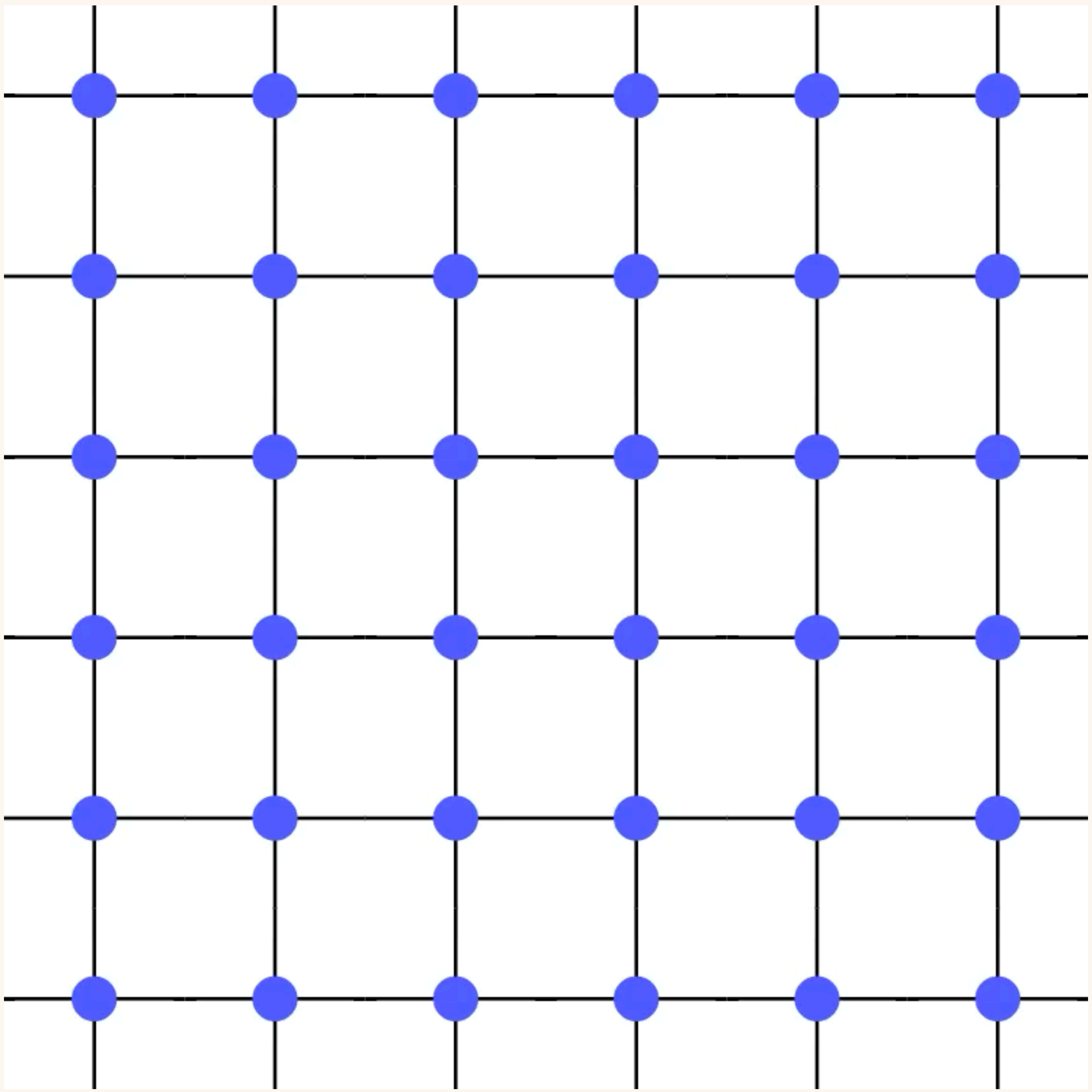


[http://www.na.scitec.kobe-u.ac.jp/~yamamoto/lectures/cse-introduction2009/cse-introduction090512.PPT]

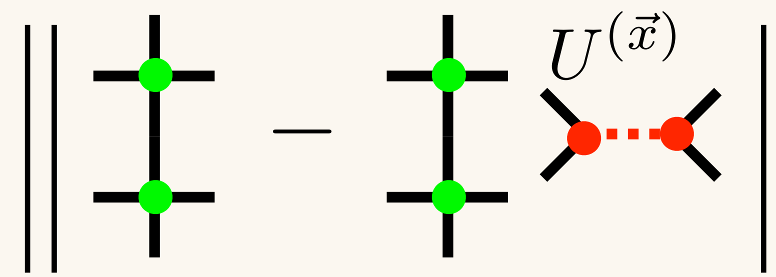
[https://smorita.github.io/TN_animation/]

◉ Cause-graining by HOTRG

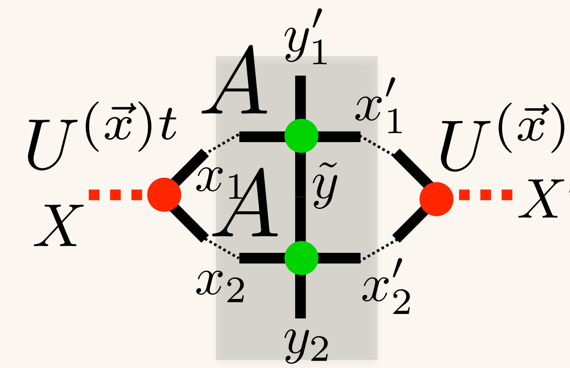
◇ Contraction by projection operator U (isometry)



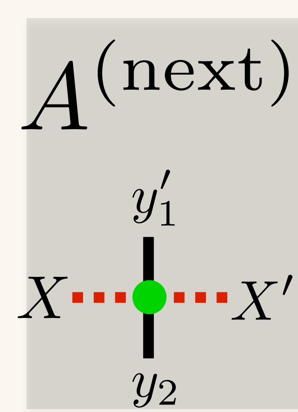
◇ Cost function



→ Find U (Isometry).

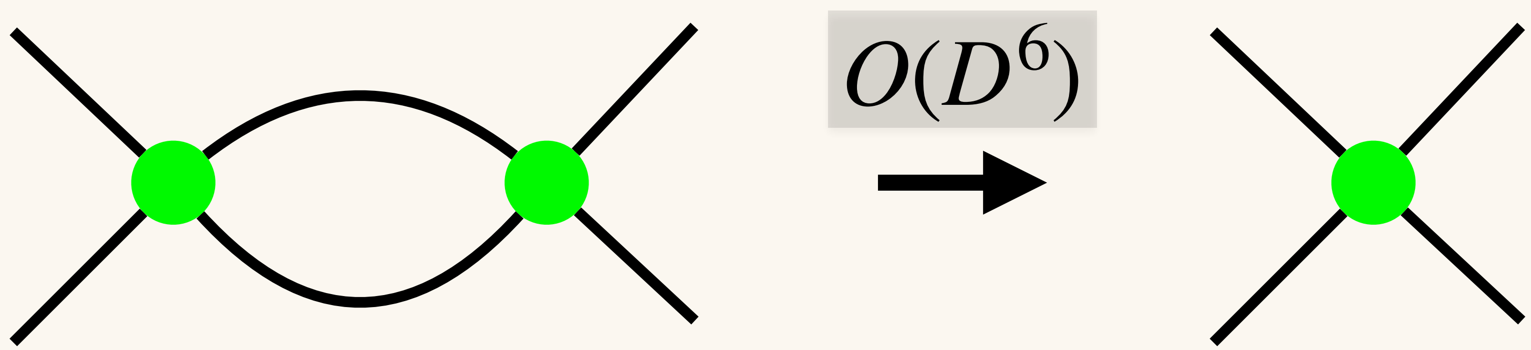


$O(D^{4\mathrm{dim}-1})$

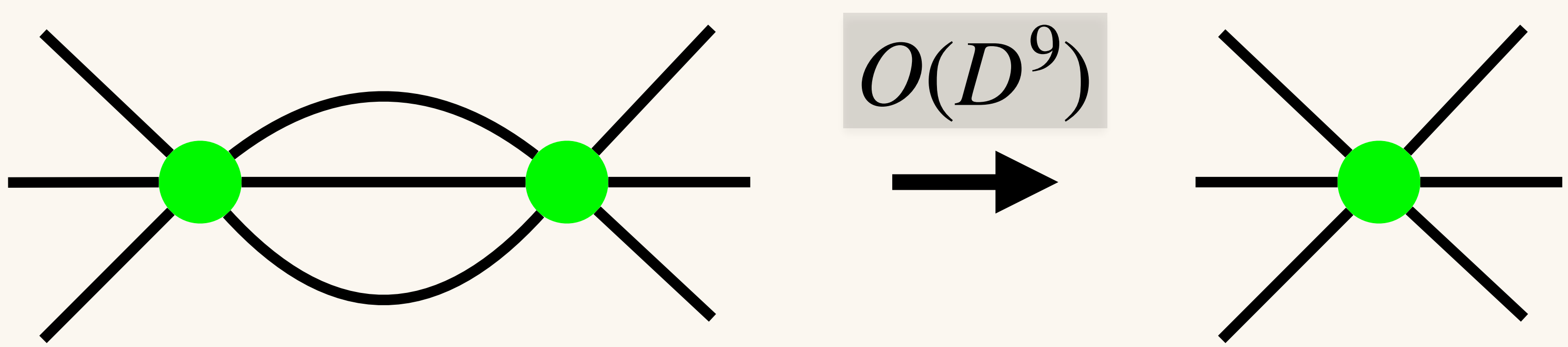


◉ Computational cost for contraction

◇ $(D^2 \times D^2)$ matrix-$(D^2 \times D^2)$ matrix product → $O(D^6)$

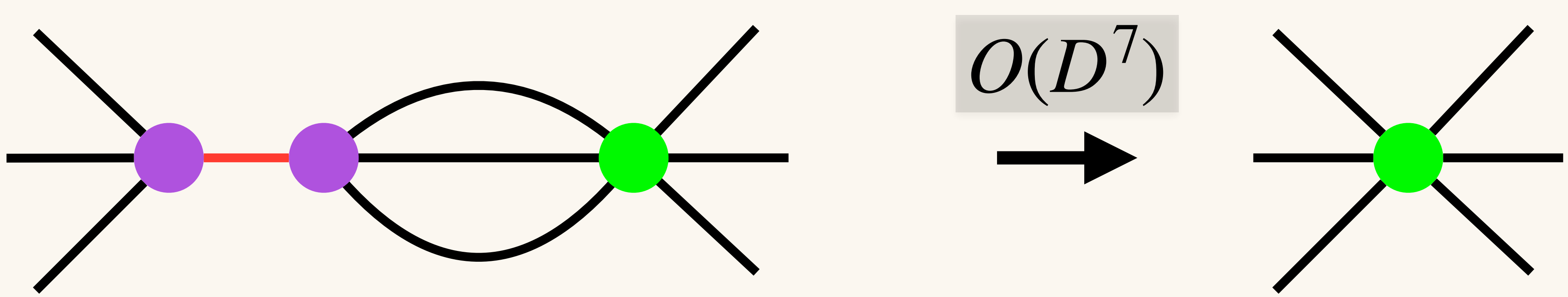


→ Higher order tensor contraction needs larger cost.



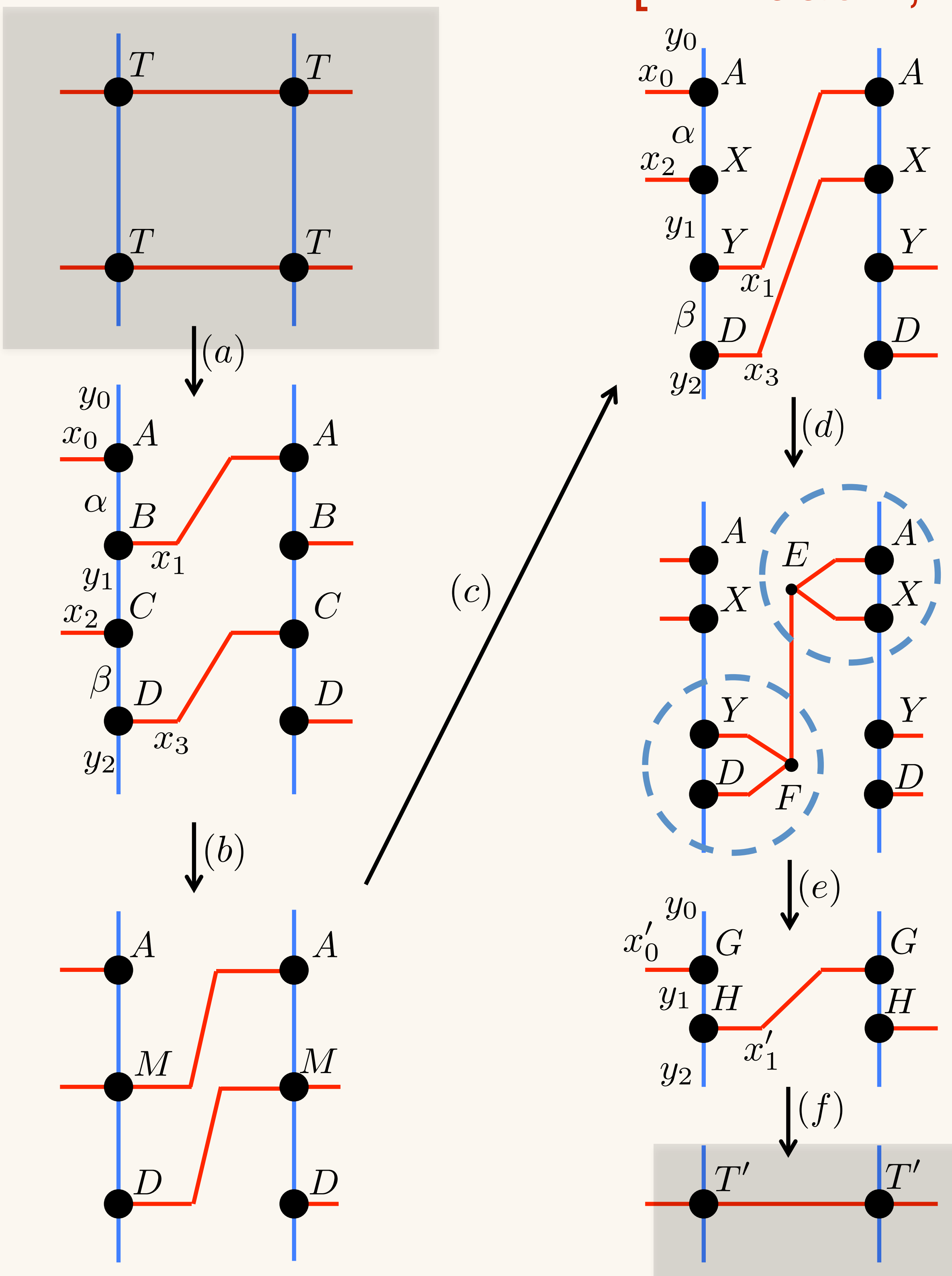
◇ Cost reduction = Low order approximation of tensor

(and/or parallel computing)



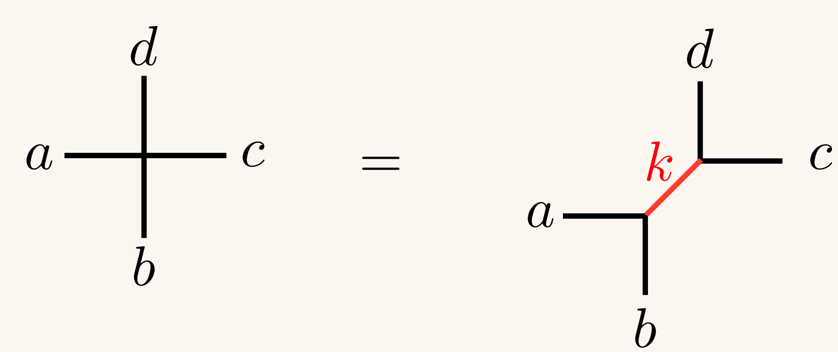
◉ Anisotropic TRG (ATRG)

[D. Adachi, T. Okubo, and S. Todo. arXiv:1906.02007]



◇ Auxiliary SVD before SVD
(In order to reduce indeces)

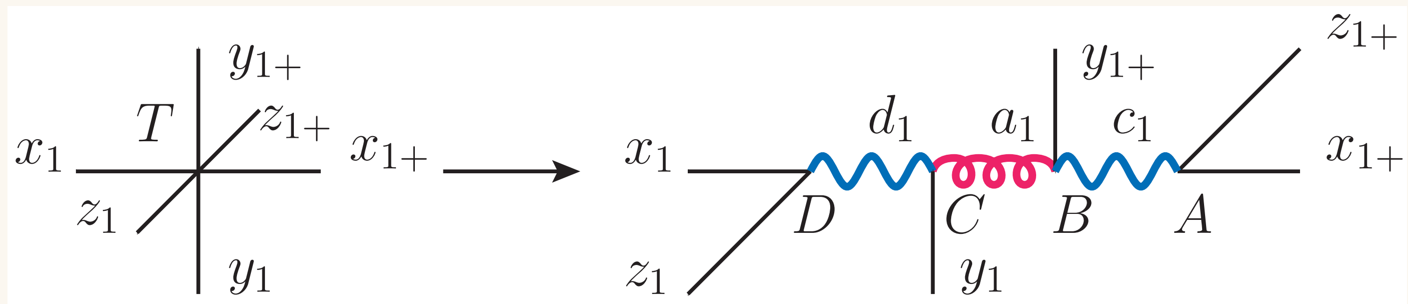
$$O(D^{4\mathrm{dim}-1}) \to O(D^{2\mathrm{dim}+1})$$



● Triad RG

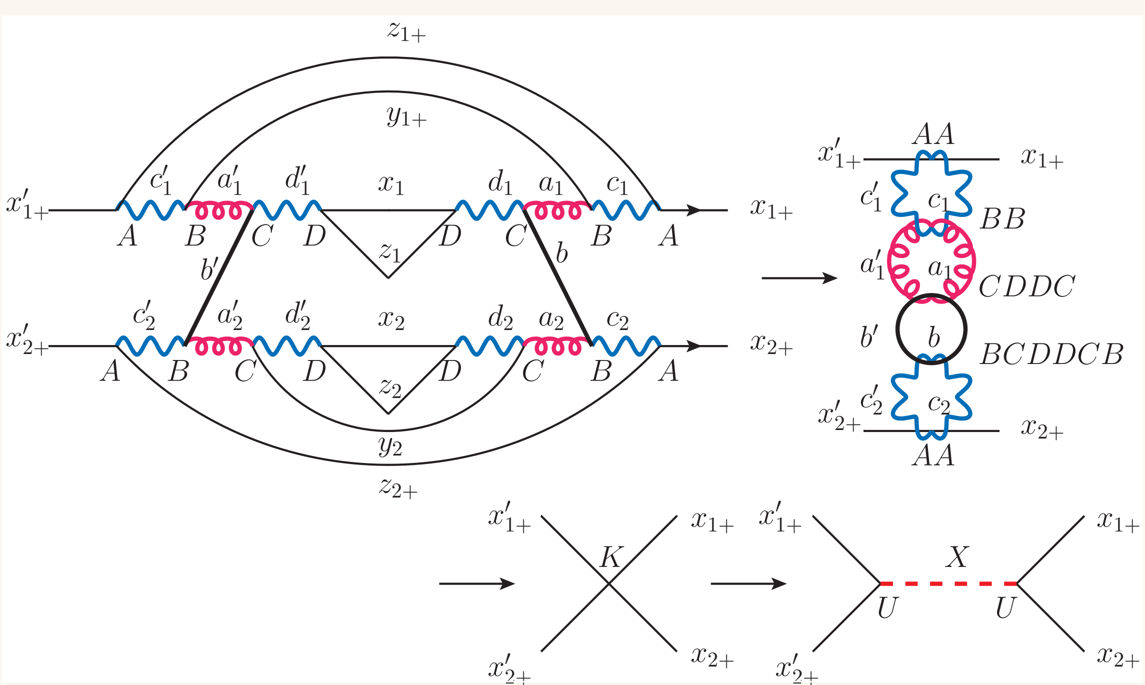
[D. Kadoh and K.N. arXiv:1912.02414]

◇ Using the Triad (Rank-3) tensor as a fundamental tensor

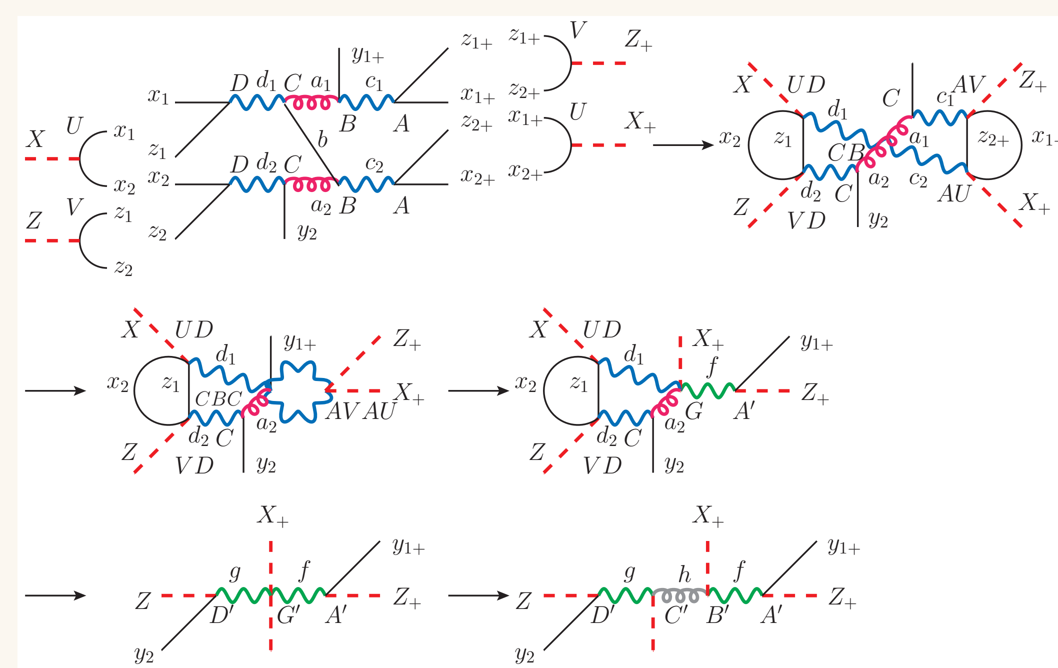


→ We apply HOTRG-like procedure to Triad tensor rep.

◇ projection operator U



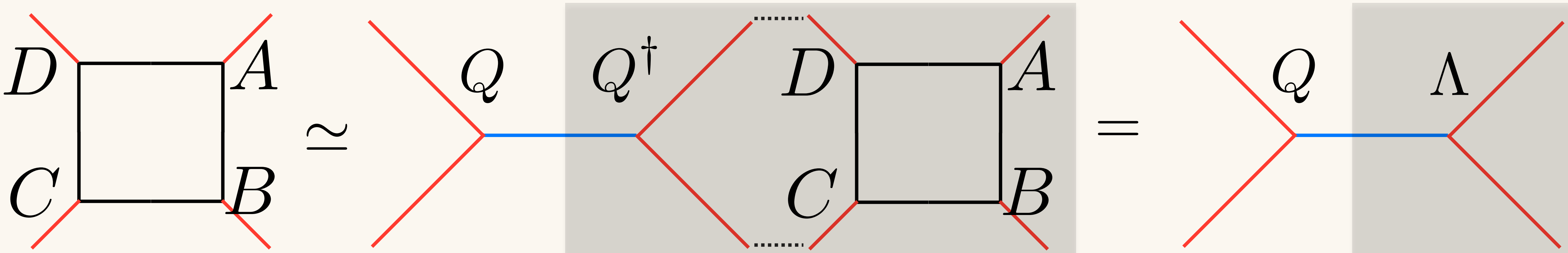
◇ contraction part



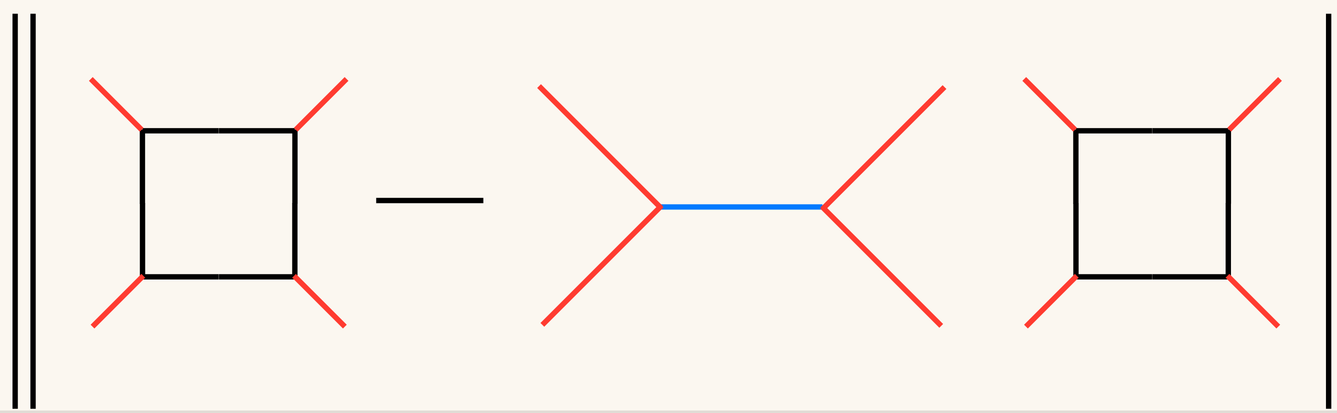
◉ Randomized-SVD

[N. Halko, et al. arXiv:0909.4061]
[S. Morita, et al. arXiv:1712.01458]

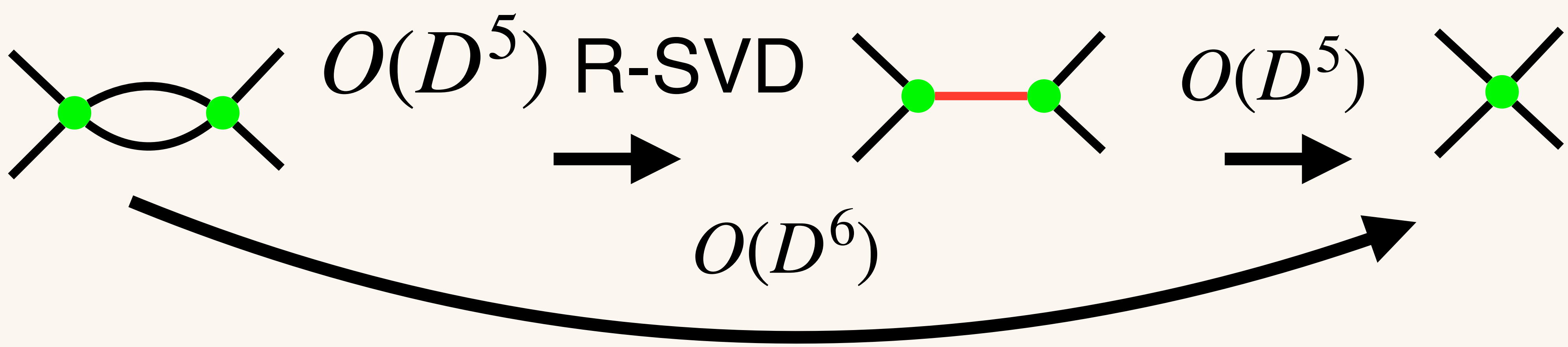
◇ Approximated contraction by orthogonal matrix Q



◇ Cost function



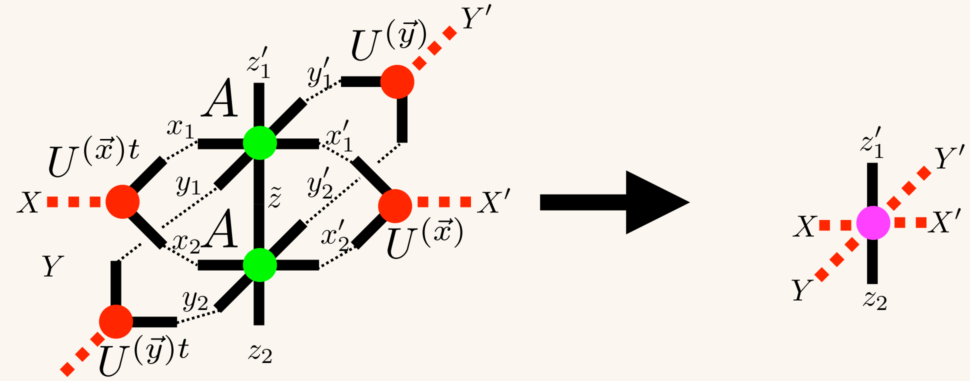
→ Approximated CONTRACTION of tensor network.



NOTE: R-SVD provides approximation of ABCD.

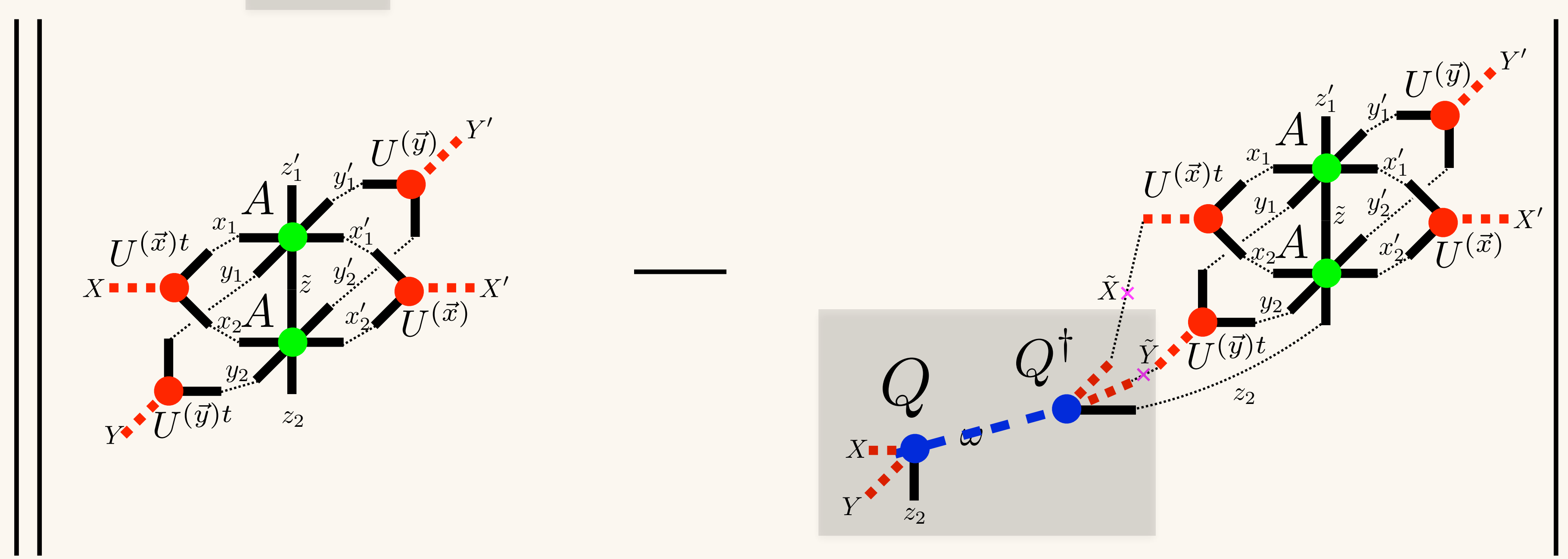
◉ HOTRG [Z.Y. Xie, J. Chen, et al. arXiv:1201.1144]

$$U^{(\vec{y})t}U^{(\vec{x})t}AAU^{(\vec{x})}U^{(\vec{y})} \to A^{(\text{next})}$$



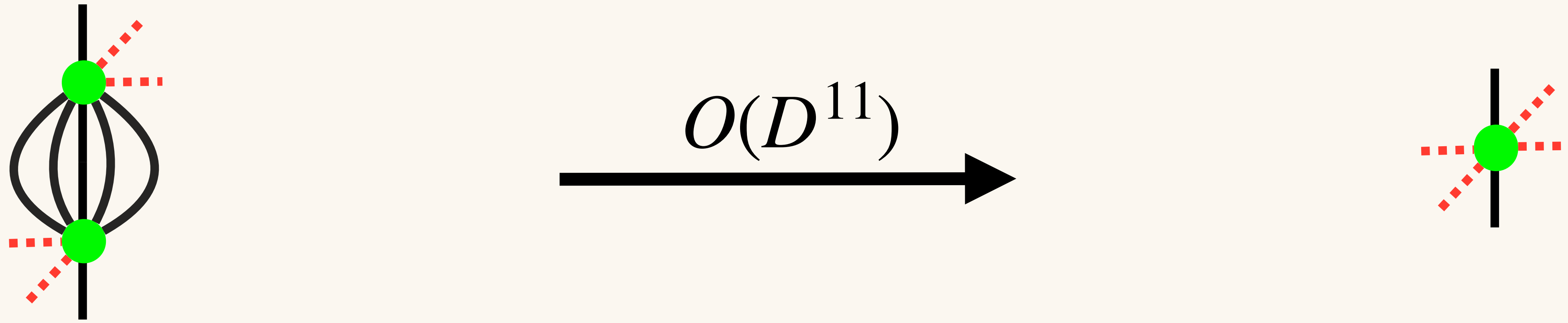
◉ HOTRG with R-SVD [K.N. arXiv:2307.14191] [D. Kadoh, K.N. arXiv:1912.02414]

$$QQ^{\dagger}U^{(\vec{y})t}U^{(\vec{x})t}AAU^{(\vec{x})}U^{(\vec{y})} \simeq A^{(\text{next})}$$

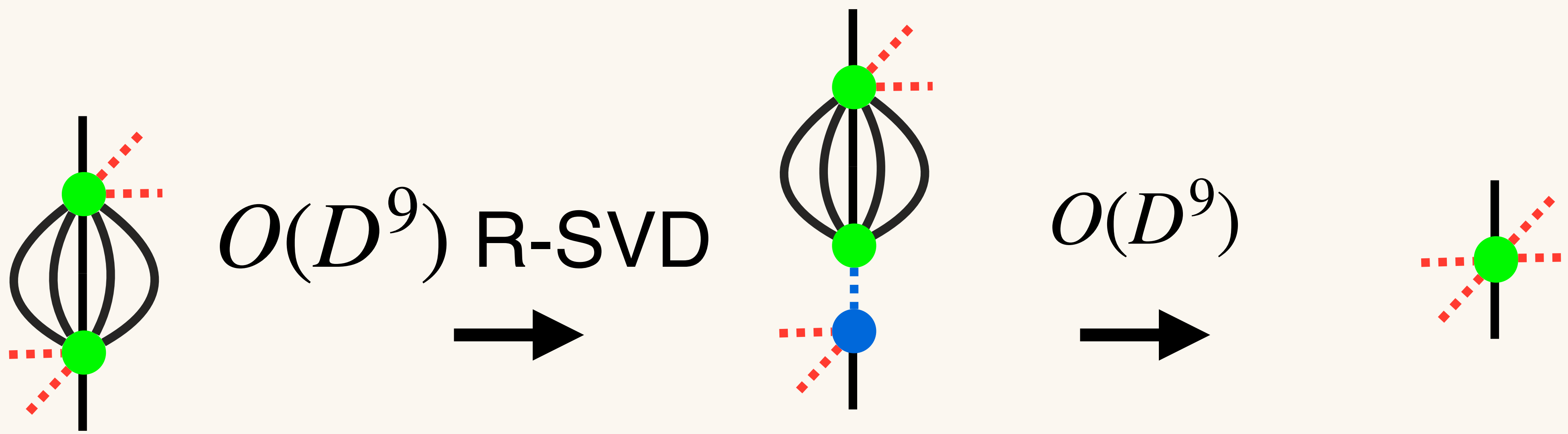


→ Contraction with R-SVD $O(D^{4d-1}) \to O(D^{3d})$

◉ HOTRG [Z.Y. Xie, J. Chen, et al. arXiv:1201.1144]



◉ HOTRG with R-SVD [K.N. arXiv:2307.14191] [D. Kadoh, K.N. arXiv:1912.02414]

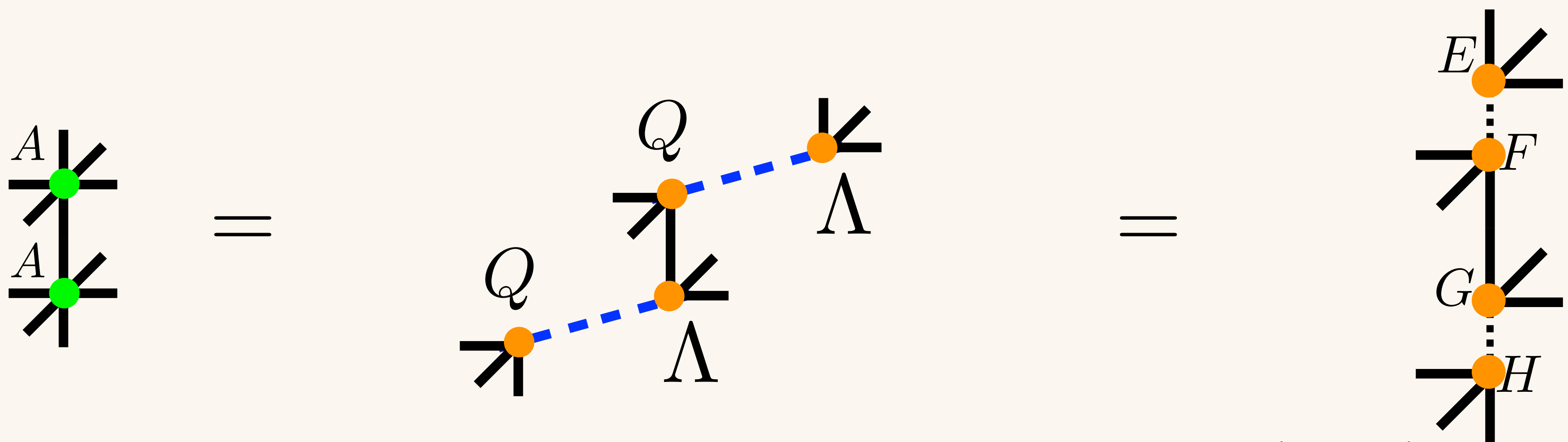


→ Contraction with R-SVD $O(D^{4d-1}) \to O(D^{3d})$

◉ Minimally-decomposed TRG(MDTRG)

[K.N. arXiv:2307.14191]

→ We already have tensor of order d+1 rep.

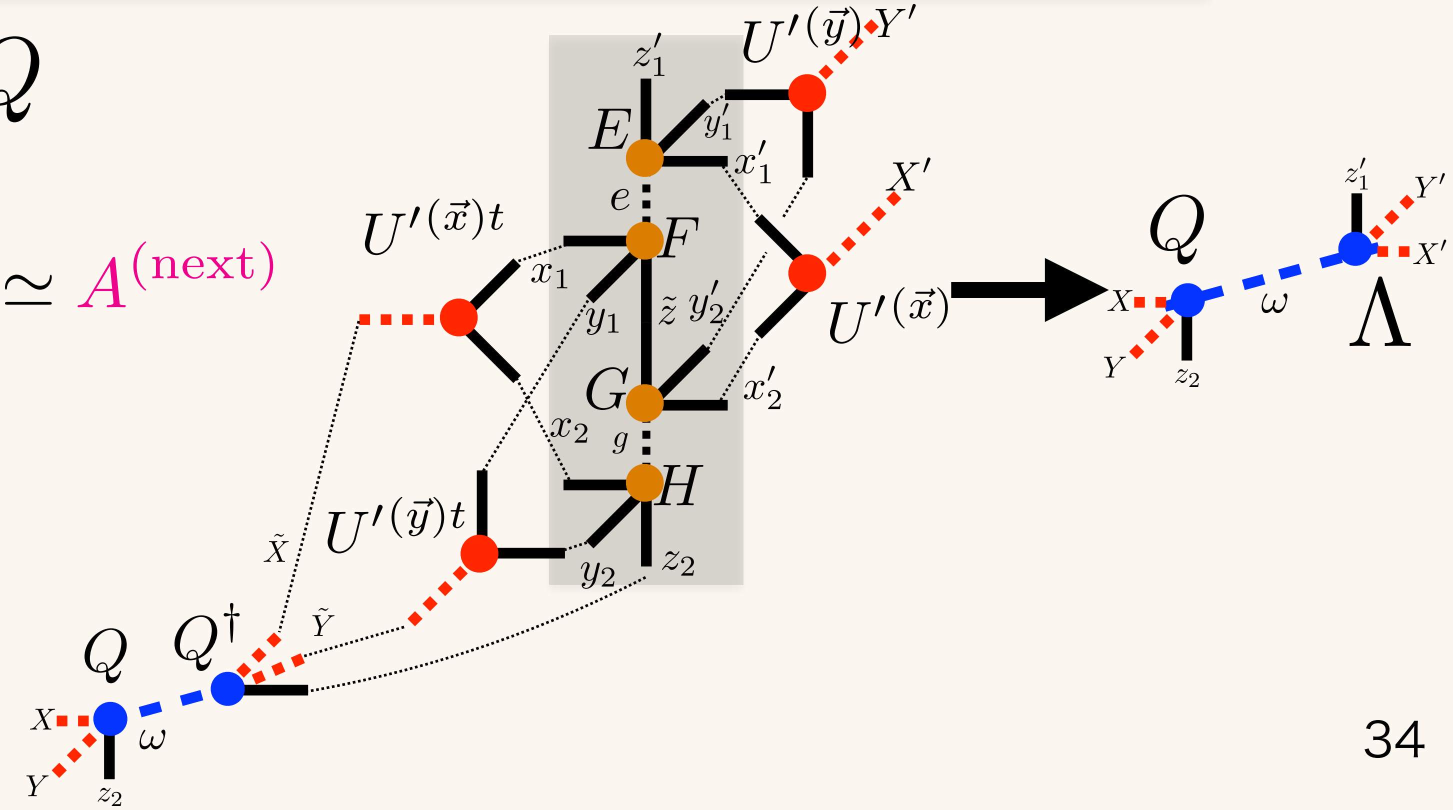


$$EFGH = Q\Lambda Q\Lambda$$

A:Order 2d   E,F,G,H:Order d+1 Q, Λ:Order d+1

◇ Contraction with EFGH, Q

$$QQ^{\dagger}U'^{(\vec{y})t}U'^{(\vec{x})t}EFGHU'^{(\vec{x})}U'^{(\vec{y})} = Q\Lambda \simeq A^{(\text{next})}$$

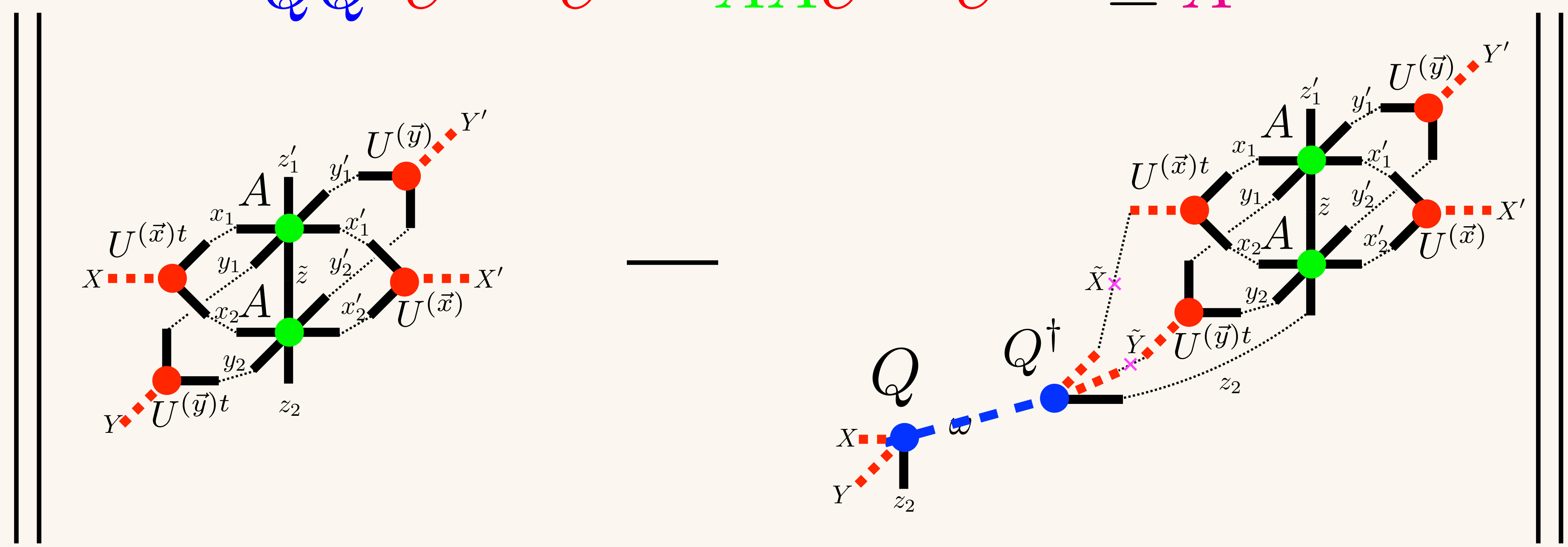


◇ Cost reduction

$$O(D^{3d}) \to O(D^{2d+1})$$

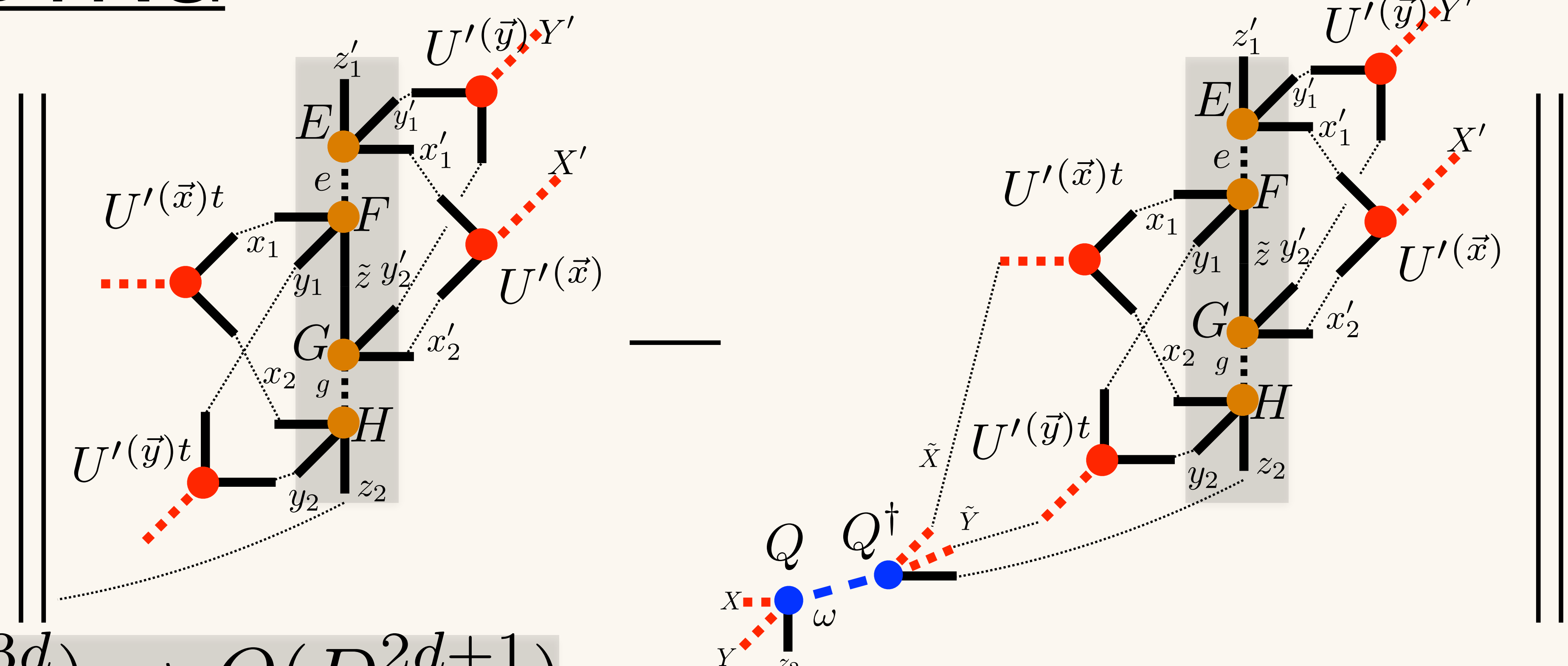
HOTRG with R-SVD

$$QQ^{\dagger}U^{(\vec{y})t}U^{(\vec{x})t}AAU^{(\vec{x})}U^{(\vec{y})} \simeq A^{(\mathrm{next})}$$



MDTRG

$$QQ^{\dagger}U'^{(\vec{y})t}U'^{(\vec{x})t}EFGHU'^{(\vec{x})}U'^{(\vec{y})} = Q\Lambda \simeq A^{(\mathrm{next})}$$

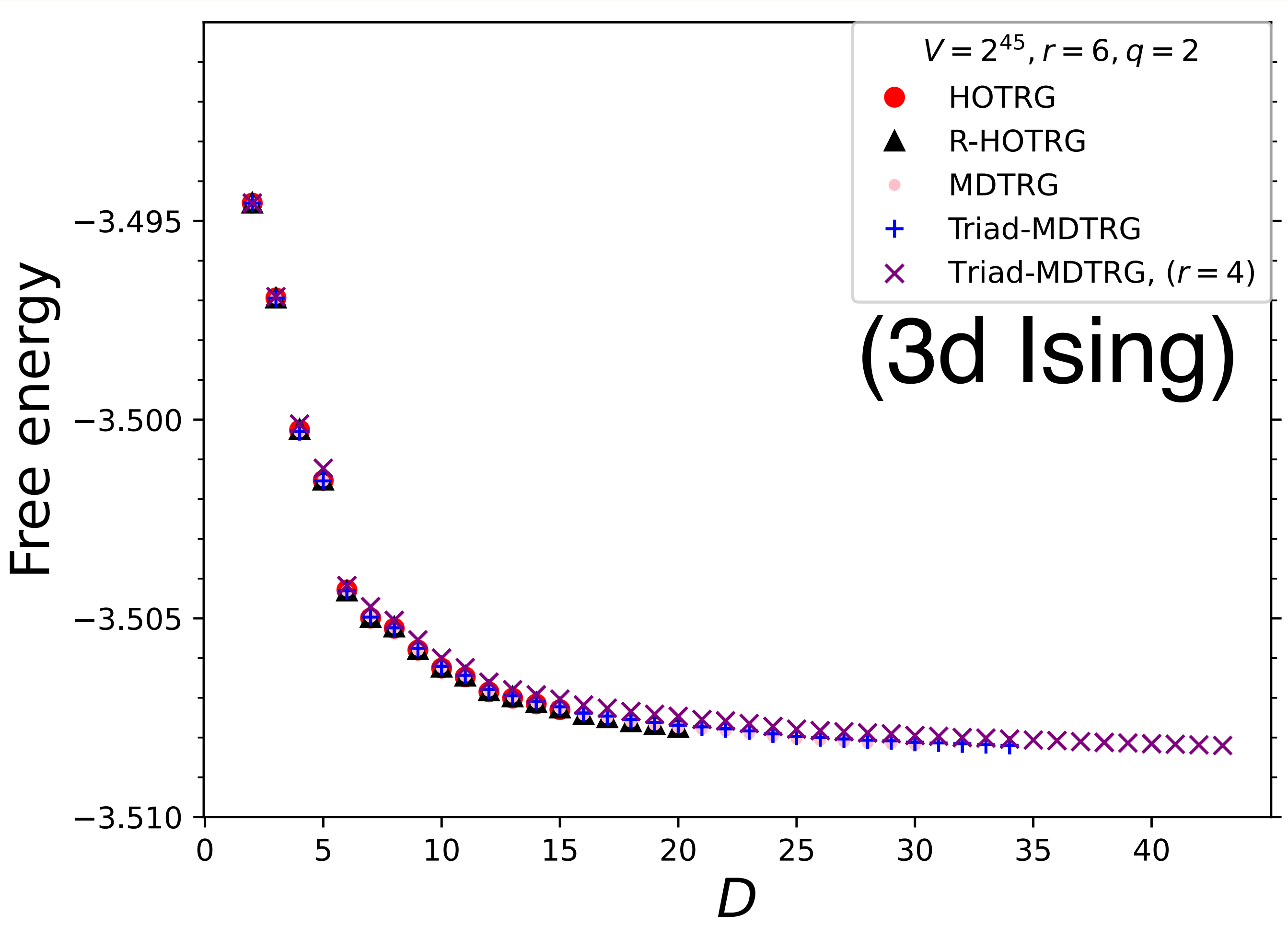


$O(D^{3d}) \to O(D^{2d+1})$

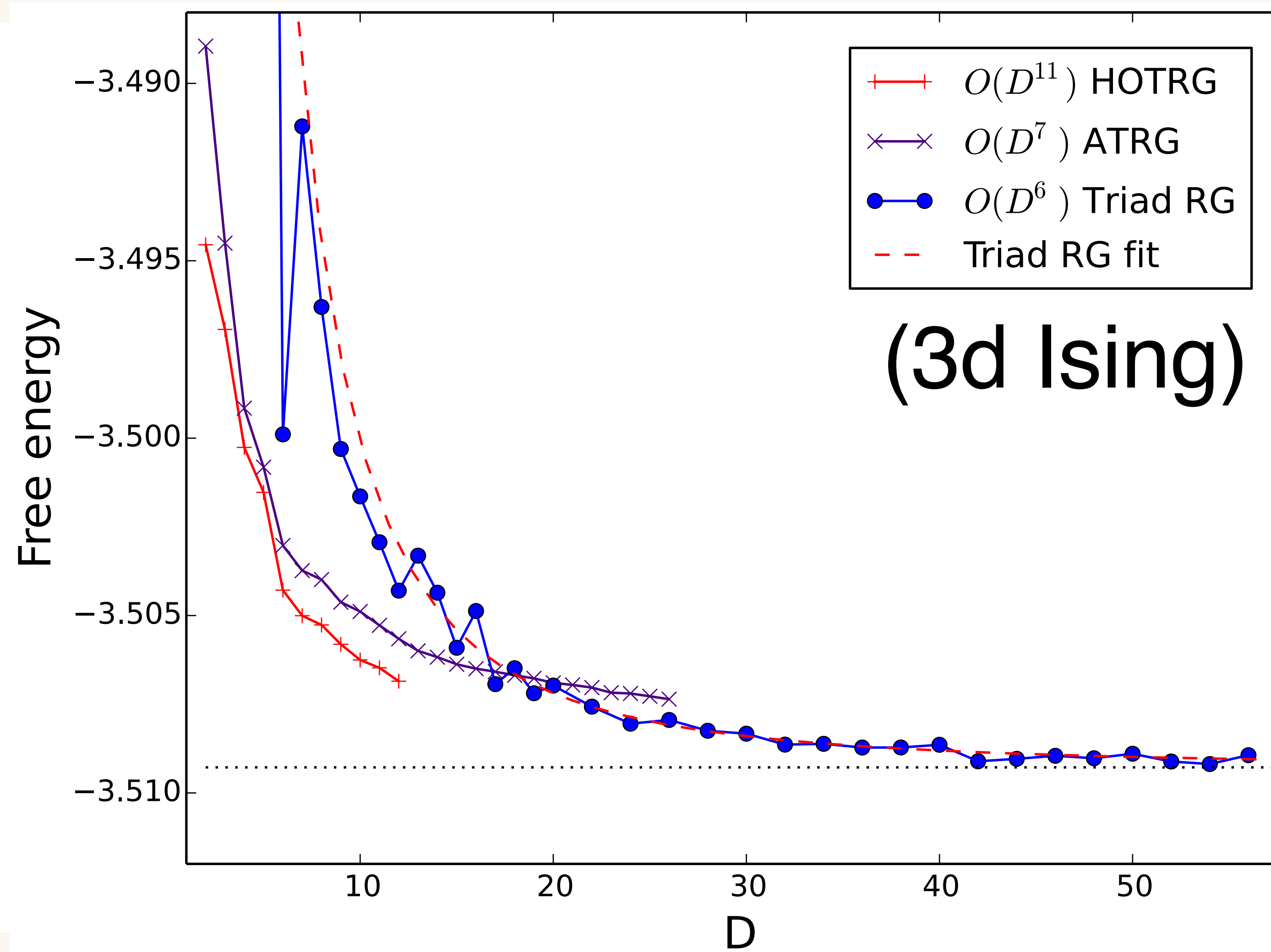
[K.N. arXiv:2307.14191]

◉ Free energy density of 3d-Ising model

[K.N. arXiv:2307.14191]



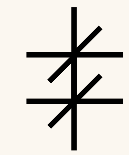
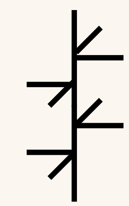
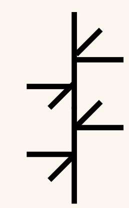
[D. Kadoh, K.N. arXiv:1912.02414]



→ R-HOTRG, MDTRG, Triad-MDTRG results are consistent with HOTRG (additional systematic error is not dominant).

◉ Summary

◇ How about HOTRG with Randomized-SVD?

	with R-SVD	w/o R-SVD	unit-cell	order
◇ HOTRG	$O(D^{3d})$	$O(D^{4d-1})$		$2d$
◇ MDTRG	$O(D^{2d+1})$	$O(D^{3d})$		$d+1$
◇ Triad-MDTRG	$O(D^{d+3})$	$O(D^{3d})$		$d+1$

◇ Can we reduce the systematic error from decomposition?

→ R-HOTRG, MDTRG, and Triad-MDTRG produce consistent result with the HOTRG. The dominant systematic error is the truncation of the isometry.

→ We have to pay attention to the cost function.