QCD thermodynamics at non-zero isospin and baryon density

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CRC-TR 211
Strong-interaction matter under extreme conditions

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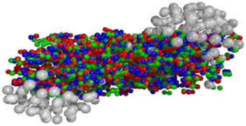
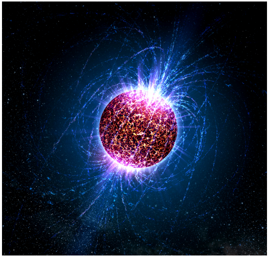
QCD systems at non-zero chemical potential

Strongly interacting matter under extreme conditions:

- a number of systems dominated by μ_B
  - Heavy-Ion collisions
  - Neutron stars

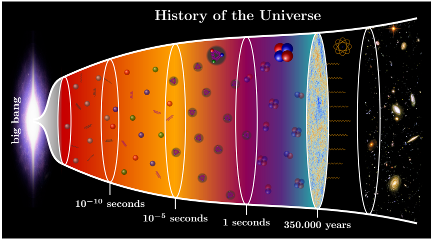
  important contribution from $\mu_Q \neq 0$
- others possibly dominated by μ_Q
  - Early Universe at large individual l_ℓ





Important for description:

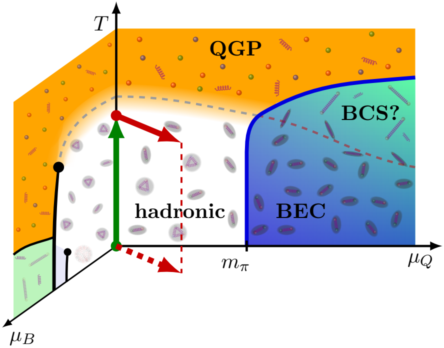
need the EoS at $\mu_B, \mu_Q, \mu_S \neq 0$



EoS at non-zero chemical potentials

Typical method: integral method with Taylor expansion around $\mu = 0$

$$p(T,\vec{\mu}) = p(T,\vec{0}) + \sum_{n_{1,2,3}=1}^{\infty} \frac{\partial^{n_1+n_2+n_3}[p(T,\vec{\mu})]}{\partial\mu_1^{n_1}\,\partial\mu_2^{n_2}\,\partial\mu_3^{n_3}}\Bigg|_{\vec{\mu}=\vec{0}} \prod_i \frac{1}{n_i!}(\mu_i)^{n_i}$$



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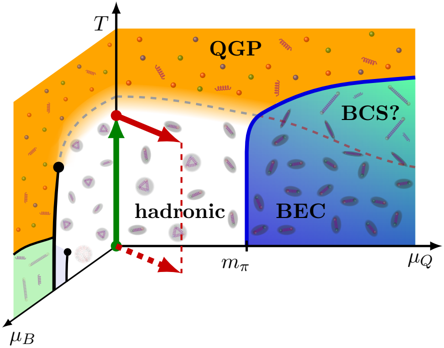
“Physical” chemical potential basis:

$$\mu_u = \frac{\mu_B}{3} + \frac{2\mu_Q}{3} \qquad \mu_d = \frac{\mu_B}{3} - \frac{\mu_Q}{3} \qquad \mu_s = \frac{\mu_B}{3} - \frac{\mu_Q}{3} - \mu_S$$



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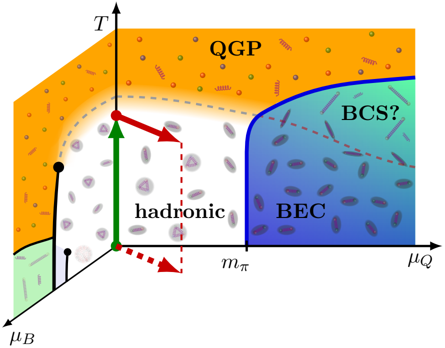
Limitations:

- finite radius of convergence (bounds difficult to compute)
- cannot extrapolate accross phase transitions



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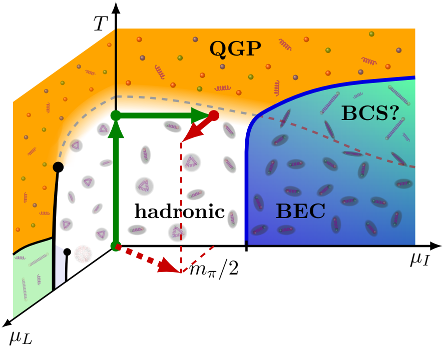
Better for systems with large μ_Q:

expand directly around non-zero μ_I

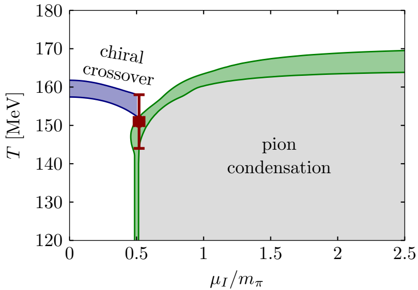


QCD at non-zero isospin chemical potential

Convenient chemical potential basis for simulations: ("isospin" basis)

$$\mu_u = \mu_L + \mu_I \qquad \mu_d = \mu_L - \mu_I \qquad \mu_s$$

$\mu_I \neq 0, \quad \mu_L = \mu_s = 0$ pure isospin chemical pot. – no sign problem



- improved actions (staggered)
- physical pion masses
- $T \neq 0$: $N_t = (6,)\,8,\ 10,\ 12$
- $T = 0$: $a = 0.22,\ 0.15$ [0.1] fm

[Brandt, Endrődi, Schmalzbauer '18]

EoS from canonical @ $T = 0$ [Detmold *et al* '12, '23, '24]

Contents

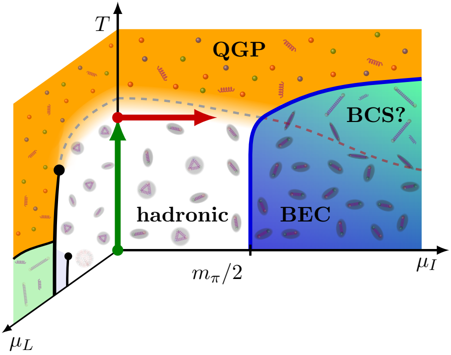
I Equation of state at non-zero μ_I

Equation of state at $\mu_I \neq 0$

$$p(T,\mu_I) = p(T,0) + \Delta p(T,\mu_I) \qquad I(T,\mu_I) = I(T,0) + \Delta I(T,\mu_I)$$

[Borsanyi *et al* '13, Bazavov *et al* '14]

to be determined



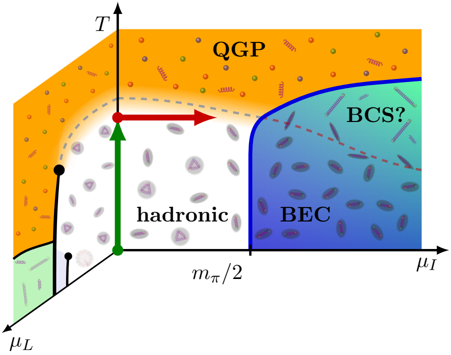
Equation of state at $\mu_I \neq 0$

$$p(T,\mu_I) = p(T,0) + \Delta p(T,\mu_I) \qquad I(T,\mu_I) = I(T,0) + \Delta I(T,\mu_I)$$

[Borsanyi *et al* '13, Bazavov *et al* '14] — to be determined

- $\Delta I(T,\mu_I) = \int_0^{\mu_I} d\mu_I' \Big(T\frac{\partial}{\partial T} - 4\Big) n_I(T,\mu_I') + \mu_I n_I(T,\mu_I)$
- $\Delta p(T,\mu_I) = \int_0^{\mu_I} d\mu \, n_I(T,\mu)$

use all spline interpolations which provide a "good" description of n_I



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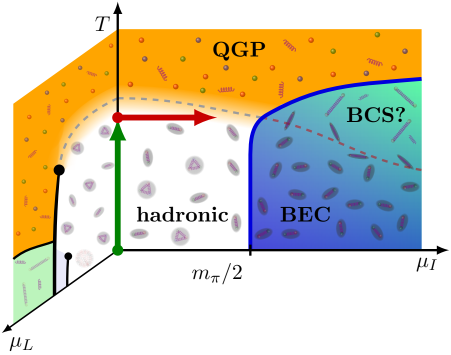
use all spline interpolations which provide a "good" description of n_I

need to compute:

$$n_I = \frac{T}{V} \mathrm{Tr}\left[M^{-1} \frac{\partial M}{\partial \mu_I} \right] \equiv \frac{T}{V} c_I$$

M: fermion matrix



Necessity of λ-extrapolations

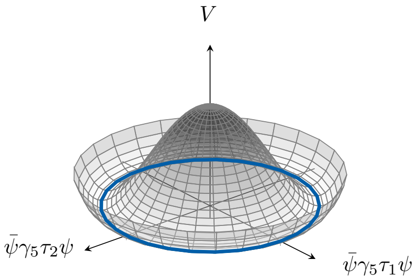
$$D = \gamma_\mu D_\mu + m_{ud} + \gamma_0 \tau_3 \, \mu_I$$

$$SU_V(2) \longrightarrow \mathrm{U}_Q(1) \longrightarrow \varnothing$$

explicit: $\mu_I \neq 0$

spontaneous: $\mu_I \geq m_\pi/2$

- cannot observe spontaneous symmetry breaking in finite V
- low mode in simulations



Necessity of λ-extrapolations

$$D = \gamma_\mu D_\mu + m_{ud} + \gamma_0 \tau_3 \, \mu_I + i\gamma_5 \tau_2 \, \lambda$$

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explicit pionic source λ

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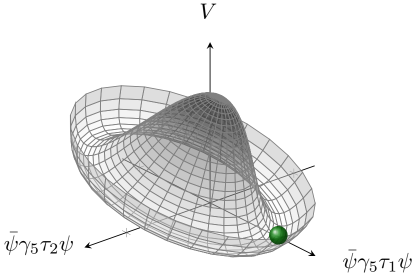

▶ need to break symmetry explicitly

→ introduce regulator: $\sim \lambda$

pionic source [Kogut, Sinclair '02]

physical results: extrapolate $\lambda \to 0$

main task for analysis

Necessity of λ-extrapolations

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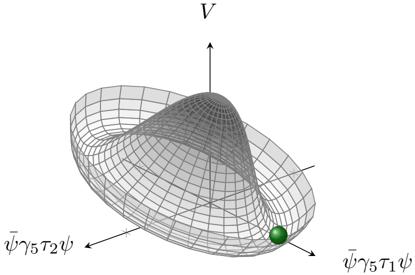

▶ need to break symmetry explicitly

➡ introduce regulator: $\sim \lambda$
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▶ facilitated by improvement program:
[Brandt, Endrődi, Schmalzbauer '18]

- leading order reweighting
- valence quark improvement for light quarks

Valence quark improvement for densities

Up to now: consider 1st derivatives – densities

$$O \equiv \mathrm{Tr}\Big[M^{-1}\hat{O}\Big] \qquad c_X\colon\;\; \hat{O} = \frac{\partial M}{\partial \mu_X} \quad \text{with} \quad M = D^\dagger(\mu)D(\mu) + \lambda^2$$

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$$\text{with} \quad \delta_O^N = \sum_{n=0}^{N} \mathcal{O}_{nn}\Big(\frac{1}{\xi_n^2+\lambda^2} - \frac{1}{\xi_n^2}\Big)$$

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CRC-TR 211

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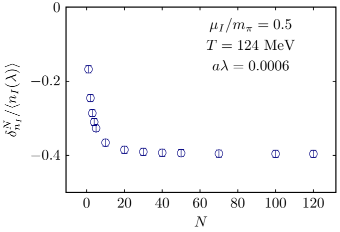

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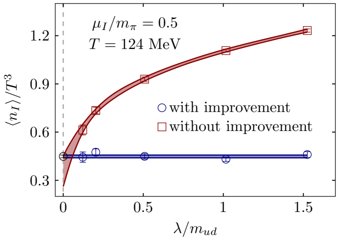
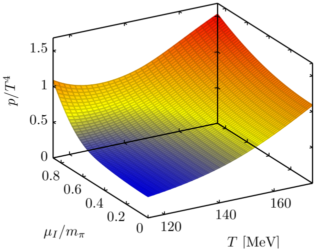
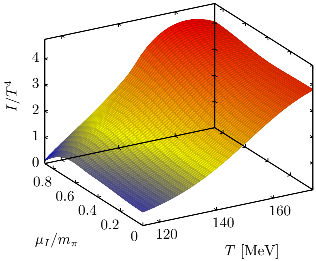
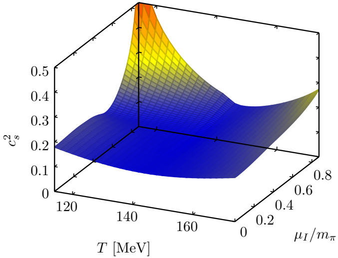


Results for the EoS at $\mu_I \neq 0$ – $N_t = 8$

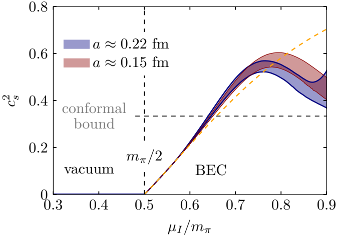




Speed of sound:



$\boldsymbol{T = 0}$

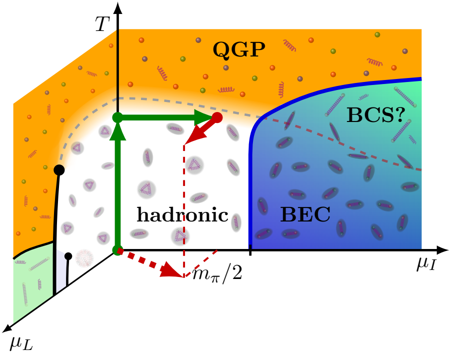


see also QC_2D: [Iida, Itou '22]

II Taylor expansion to non-zero $\mu_{L,s}$

Extension to μ_L, $\mu_s \neq 0$:

$$p(T,\vec{\mu}) = p(T,\mu_I,0,0) + \sum_{n,m=1}^{\infty} \frac{1}{n!\,m!} \frac{\partial^n \partial^m [p(T,\vec{\mu})]}{\partial \mu_L^n \, \partial \mu_s^m}\bigg|_{\mu_L,\mu_s=0} (\mu_L)^n (\mu_s)^m$$



Taylor expansion around $\mu_I \neq 0$

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- Leading order: only non-zero coefficients are

$$\chi_2^L(T,\mu_I) \equiv \left. \frac{\partial^2 [p(T,\vec{\mu})]}{\partial \mu_L^2} \right|_{\mu_L,\mu_s=0} \quad \& \quad \chi_2^s(T,\mu_I) \equiv \left. \frac{\partial^2 [p(T,\vec{\mu})]}{\partial \mu_s^2} \right|_{\mu_L,\mu_s=0}$$

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- **Generically:** $$\chi_2^X = \frac{T}{V}\Big[\underbrace{\langle c_{XX}\rangle}_{\text{connected}} + \underbrace{\langle (c_X)^2\rangle - \langle c_X\rangle^2}_{\text{disconnected}}\Big]$$

with $$c_X = \mathrm{Tr}\Big[M^{-1}\frac{\partial M}{\partial\mu_X}\Big]$$ M: fermion matrix

$$c_{XX} = \frac{\partial c_X}{\partial\mu_X} = \mathrm{Tr}\Big[M^{-1}\frac{\partial^2 M}{(\partial\mu_X)^2}\Big] + \mathrm{Tr}\Big[M^{-1}\frac{\partial M}{\partial\mu_X}M^{-1}\frac{\partial M}{\partial\mu_X}\Big]$$

Valence quark improvement for susceptibilities

$$\chi_2^X = \frac{T}{V}\left[\langle c_{XX}\rangle + \langle (c_X)^2\rangle - \langle c_X\rangle^2\right]$$

- disconnected terms: as above $\langle (c_X - \delta^N_{c_X})^2\rangle - \langle c_X - \delta^N_{c_X}\rangle^2$

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- two connected terms:

  $$c_{XX} = \mathrm{Tr}\Big[M^{-1}\frac{\partial^2 M}{(\partial\mu_X)^2}\Big] + \mathrm{Tr}\Big[M^{-1}\frac{\partial M}{\partial\mu_X}M^{-1}\frac{\partial M}{\partial\mu_X}\Big]$$

  - treat as above

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$$c_{XX} = \mathrm{Tr}\left[M^{-1}\frac{\partial^2 M}{(\partial\mu_X)^2}\right] + \mathrm{Tr}\left[M^{-1}\frac{\partial M}{\partial\mu_X}M^{-1}\frac{\partial M}{\partial\mu_X}\right]$$

  - treat as above

$$C_{12} \equiv \mathrm{Tr}\left[M^{-1}\hat{O}^{(1)}M^{-1}\hat{O}^{(2)}\right] \approx \left\langle \sum_{n,m=0}^{N} \frac{\mathcal{O}^{(1)}_{nm}}{\xi_m^2+\lambda^2}\frac{\mathcal{O}^{(2)}_{nm}}{\xi_n^2+\lambda^2}\right\rangle$$

Valence quark improvement for susceptibilities

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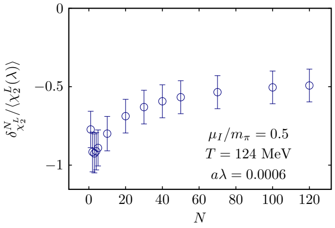

→ improvement term:

$$\delta^N_{C_{12}} = \sum_{n,m=1}^{N} \mathcal{O}^{(1)}_{nm}\mathcal{O}^{(2)}_{mn}\left(\frac{1}{\xi_n^2+\lambda^2}\frac{1}{\xi_m^2+\lambda^2} - \frac{1}{\xi_n^2}\frac{1}{\xi_m^2}\right)$$

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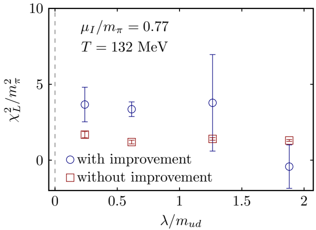

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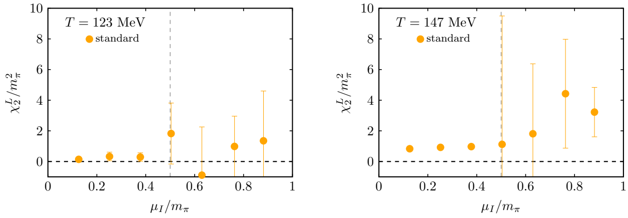
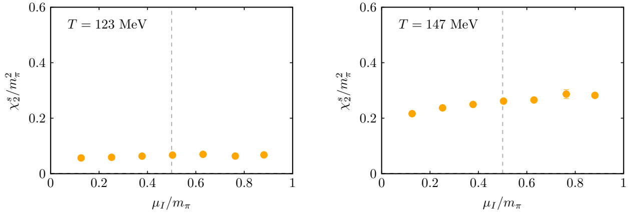
λ-extrapolated Taylor coefficents @ $N_t = 8$

- results for $\chi_2^L(T,\mu_I)$ using standard λ-extrapolations:



Large uncertainties for $\chi_2^L(T,\mu_I)$ in the BEC phase!

- $\chi_2^s(T,\mu_I)$ not affected (no source parameter)



Improved computation method

Observation: equal connected parts in μ_L and μ_I derivatives

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- for EoS computation: model independent spline interpolation of n_I

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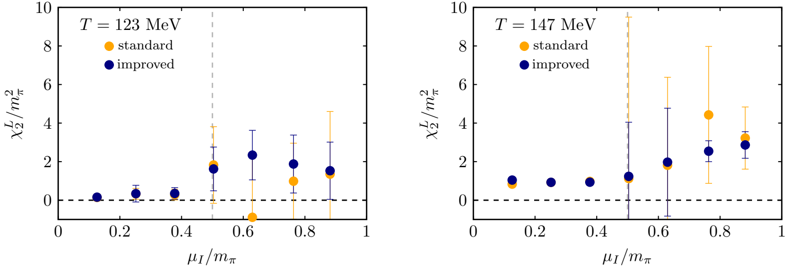
- finally compute $\chi_2^L(T,\mu_I)$ using

$$\chi_2^L(T,\mu_I) = \underbrace{\chi_2^I(T,\mu_I)}_{\lambda=0} + \frac{T}{V}\Big[\underbrace{\big\langle (c_L)^2\big\rangle - \big\langle c_L\big\rangle^2 - \big\{\big\langle (c_I)^2\big\rangle - \big\langle c_I\big\rangle^2\big\}}_{\lambda\neq 0}\Big]$$

(λ-extrapolation for disconnected typically better behaved)

λ-extrapolated Taylor coefficents @ $N_t = 8$

- comparison of the methods for $\chi_2^L(T, \mu_I)$ in BEC:



→ reduced uncertainties for $\chi_2^L(T, \mu_I)$ in the BEC phase!

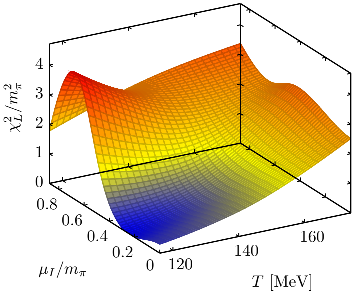
Results for Taylor coefficients @ $N_t = 8$

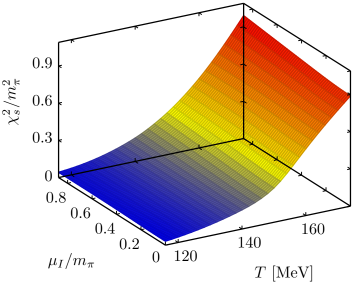
To extend the EoS: interpolate Taylor expansion coefficients

as for EoS: [Brandt, Cuteri, Endrődi '22] [Brandt, Endrődi '16]

use spline interpolations with Monte-Carlo generated nodepoints

- constraint: $\chi_2^L(T, \mu_I)$ & $\chi_2^s(T, \mu_I)$ vanish at $T = 0$





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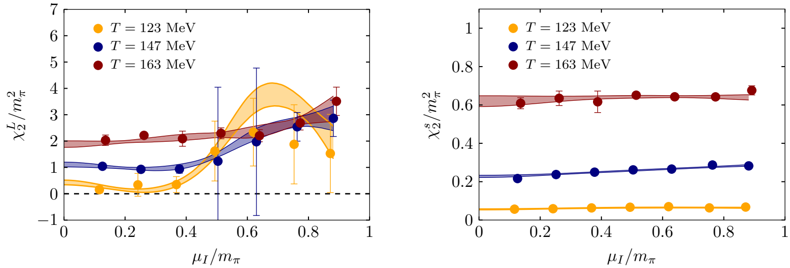


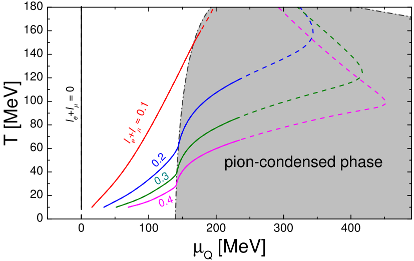
III An application: EoS at non-zero charge chemical potential

early Universe: sum of lepton flavour asymmetries is constrained

$|l_e + l_\mu + l_\tau| < 0.012$ [Oldengott, Schwarz '17]

but: for large individual l_ℓ

→ large μ_Q and small μ_B & μ_S along trajectories



- model equation of state – matched to lattice at pure μ_I

  [Vovchenko, *et al* '20]

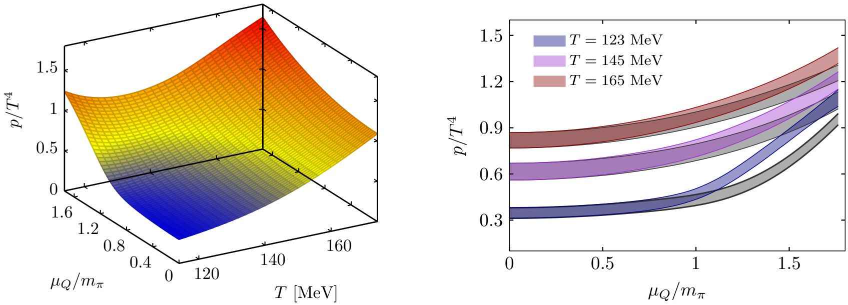
- Taylor expansion from $\vec{\mu} = 0$ – no pion condensation

  [Middeldorf-Wygas, *et al* '20]

- can provide a full lattice EoS

non-zero μ_Q in the isospin basis:

$$\mu_I = \frac{\mu_Q}{2} \qquad \mu_L = \frac{\mu_Q}{6} \qquad \mu_s = -\frac{\mu_Q}{3}$$



assumption: BEC phase boundary does not change drastically

- Simulations at $\mu_I \neq 0$:
  offer a novel expansion point to explore (μ_B, μ_Q, μ_S) space
- λ-extrapolations necessary:
  ➡ large uncertainties for $\chi_2^L(T, \mu_I)$ in BEC phase
- alternative: computation via $\chi_2^I(T, \mu_I)$
  obtained from spline interpolation of n_I
  ➡ improves uncertainties
  some details still to be understod
- application and outlook:
  compute EoS at non-zero μ_Q
  and in its vicinity