

Exercises

CRC 1660 Annual Graduate School Sep. 22–26, 2025, Schwäbisch Gmünd
Introduction to Effective Field Theories

This is a collection of exercises which accompany the lecture. The exercises vary in difficulty and length. You will probably not be able to solve all exercises during the exercise session. You may use mathematica for analytical/numerical calculations where appropriate. Discussions are strongly encouraged.

1 Dimensional analysis I: Planck units

Construct the Planck length l_P , the Planck mass m_P , and the Planck time t_P from the fundamental constants G , $\hbar$ and c . Estimate their size in SI units and compare to typical time scales from particle physics. What is the significance of these units?

2 Dimensional analysis II: photon-photon scattering

Consider photon-photon scattering using the Euler-Heisenberg Lagrangian,

$$\mathcal{L}_{eff} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + c_1(F_{\mu\nu}F^{\mu\nu})^2 + c_2(F_{\mu\nu}\tilde{F}^{\mu\nu})^2 + \dots,$$

discussed in the lecture. Determine the dependence of the LEC's c_1 and c_2 on the elementary charge e and the electron mass m_e from the leading QED diagram contributing to this process. Estimate the size of the amplitude for photon-photon scattering at energies well below the electron mass, $E_\gamma \ll m_e$. Estimate the size of the cross section for $E_\gamma \ll m_e$.

3 Naturalness for a square well potential

Consider the attractive radially symmetric square well potential

$$\begin{aligned} V(r) &= -V_0, & r < r_0, \\ &= 0, & r \geq r_0, \end{aligned}$$

where $V_0 > 0$. The S -wave scattering length for scattering of a particle of mass m off this potential is

$$a = r_0 \left(1 - \frac{\tan(\kappa_0 r_0)}{\kappa_0 r_0} \right), \quad (1)$$

where $\kappa_0 = \sqrt{mV_0}$. Assume a constant probability distribution $P(\kappa_0)$ for κ_0 (r_0 is fixed) and derive the resulting probability distribution $P(a)$ for a . Use the relation

$$P(a)da = P(\kappa_0)d\kappa_0.$$

For $\kappa_0 r_0 \gg \pi$ the term $\tan(\kappa_0 r_0)$ in Eq. (1) changes for small variations of $\kappa_0 r_0$ much more strongly than $1/\kappa_0 r_0$. The variation of $1/\kappa_0 r_0$ can thus be neglected. Sketch the result and give an interpretation. What the most probable value of a ?

4 Power Counting

Consider a short-range effective field theory with natural interaction terms. In the lecture it was shown that a Feynman diagram with arbitrary 2-particle exchanges for small momenta k scales like k^ν . The following expression was derived for the exponent ν :

$$\nu = 3L + 2 + \sum_i (2i - 2)V_{2i},$$

where L denotes the number of loops and V_{2i} the number of 2-body interactions with $2i$ derivatives. Generalize this expression to include arbitrary natural N -particle interactions. Give the diagrams for $3 \rightarrow 3$ particle scattering in the lowest three orders in ν .

5 Dimer formalism I

In systems with short-range interactions and a large scattering length, it is often convenient to introduce an explicit dimer field. Show that the Lagrangian with the dimer field d ,

$$\mathcal{L}' = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi + \frac{g_2}{4} d^\dagger d - \frac{g_2}{4} (d^\dagger \psi^2 + (\psi^\dagger)^2 d) - \frac{g_3}{36} d^\dagger d \psi^\dagger \psi + \dots, \quad (2)$$

is equivalent to the standard Lagrangian without explicit dimer field

$$\mathcal{L} = \psi^\dagger \left(i\partial_t + \frac{\nabla^2}{2m} \right) \psi - \frac{g_2}{4} (\psi^\dagger \psi)^2 - \frac{g_3}{36} (\psi^\dagger \psi)^3 + \dots,$$

up to N -body interactions with $N \geq 4$. Use the classical equation of motion for the dimer field. Do you obtain the same observables in 3- and 4-body systems with the two Lagrangians?

6 Dimer formalism II

Consider the two-particle problem in the theory with an explicit dimer field using the Lagrangian in Eq. (2).

1. Which diagrams contribute to the full dimeron propagator? How is the 2-body scattering amplitude obtained from the full dimer propagator?
2. Calculate the full dimeron propagator and the 2-body scattering amplitude.