1 Dimensional Analysis I: Planck units

Construct Planck length, mass, and time out of G, $\hbar$, c

$[G] = \frac{m^3}{kg\, s^2}$ Newton's constant

Aside: $\frac{kg\, m}{s^2} = [G] \frac{(kg)^2}{m^2}$

$F = G \frac{m_1 m_2}{r^2}$

$[\hbar] = Js = \frac{kg\, m^2}{s}$

$[c] = \frac{m}{s}$

$\hookrightarrow \quad m_p = \left(\frac{\hbar c}{G}\right)^{1/2} \approx 2.2 \cdot 10^{-8}\ kg$

Note: $m_p \approx 1.7 \cdot 10^{-27}\ kg$

$m_e \approx 9.1 \cdot 10^{-31}\ kg$

$\hookrightarrow \quad l_p = \left(\frac{\hbar G}{c^3}\right)^{1/2} \approx 1.6 \cdot 10^{-35}\ m$

Note: $r_e < 10^{-18}\ m$

$r_p \approx 10^{-15}\ m$

$\hookrightarrow \quad t_p = \left(\frac{\hbar G}{c^5}\right)^{1/2} \approx 5.4 \cdot 10^{-44}\ s$

Note: $\tau_\Delta \approx 5.6 \cdot 10^{-24}\ s$ strong decay

$\left.\begin{array}{l} \tau_\mu \approx 2 \cdot 10^{-6}\ s \\ \tau_{\pi^\pm} \approx 2.6 \cdot 10^{-8}\ s \end{array}\right\}$ weak decay

$\left.\begin{array}{l} \tau_{\pi^0} \approx 8 \cdot 10^{-17}\ s \\ \tau_\eta \approx 5 \cdot 10^{-19}\ s \end{array}\right\}$ EM decay

2 Dimensional analysis II : photon-photon scattering

$$\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + c_1 (F_{\mu\nu} F^{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2 + \ldots$$

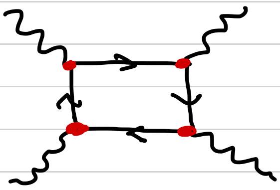
Dimensional analysis: $[\mathcal{L}] = 4$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \longrightarrow [A_\mu] = 1, [\partial_\mu = 1]$

$\hookrightarrow [F_{\mu\nu}] = 2$, $[\tilde{F}_{\mu\nu}] = [\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} F^{\alpha\beta}] = 2$

Therefore we have $[c_1] = [c_2] = -4$

Leading QED contribution:



$\sim \mathcal{M}$

- m_e is only mass scale
- 4 couplings $\rightarrow$ 4 powers of e

$\hookrightarrow c_1 (F_{\mu\nu} F^{\mu\nu})^2 + c_2 (F_{\mu\nu} \tilde{F}^{\mu\nu})^2$

$\hookrightarrow \quad c_1 \sim c_2 \sim \frac{e^4}{m_e^4}$

Amplitude for $E_\gamma \ll m_e$: $\quad \partial_\mu \rightarrow (p_\gamma)_\mu \sim E_\gamma$

$\hookrightarrow \quad \mathcal{M} \sim \frac{e^4}{m_e^4} E_\gamma^4$

$\gamma\gamma$ cross section for $E_\gamma \ll m_e$

$\int \left(\frac{d^3k}{2E}\right)^2 \delta^4(\ldots)$

$\hookrightarrow \quad d\sigma = \frac{1}{|j|} |\mathcal{M}|^2 \int dLIPS$

$$\sim \frac{1}{E_\gamma^2} \left(\frac{e^4}{m_e^4} E_\gamma^4\right)^2 (E_\gamma)^0 \sim \frac{e^8}{m_e^8} E_\gamma^6$$

alternatively from $c_{1,2} \sim \frac{e^4}{m_e^4}$ and insight that correct dimension can only come from E_γ

3 Naturalness for square well potential

$r_0 = \text{const.} \longrightarrow \text{take} \quad x = \kappa_0 r_0$

$$\hookrightarrow \quad a/r_0 = 1 - \frac{\tan(x)}{x}$$

$$P(a)\,da = \underbrace{P(x)}_{\to 1}\,dx = \left|\frac{dx}{da}\right| da = \frac{1}{|da/dx|}\,da$$

normalize later

$$(\tan x)' = 1 + \tan^2 x$$

neglect

$$\hookrightarrow \quad \frac{da}{dx} = r_0 \frac{d}{dx}\left(1 - \frac{\tan x}{x}\right) = r_0\left(\cancel{\frac{\tan x}{x^2}} - \frac{1+\tan^2 x}{x}\right)$$

$$= r_0\left(-\frac{1}{x} - x\left(1 - \frac{a}{r_0}\right)^2\right) = -r_0 x\left(\frac{1}{x^2} + \left(1 - \frac{a}{r_0}\right)^2\right)$$

$$\hookrightarrow \quad P(a)\,da = \frac{da}{r_0^2 \kappa_0}\,\mathcal{N}\left(\frac{1}{\kappa_0^2 r_0^2} + \left(1 - \frac{a}{r_0}\right)^2\right)^{-2} = \frac{1}{\kappa_0}\,\frac{\mathcal{N}\,da}{1/\kappa_0^2 + (a - r_0)^2}$$

$x = \kappa_0 r_0$ normalization

Normalization: $\int_{-\infty}^{\infty} P(a)\,da = 1 \longrightarrow \mathcal{N} = \frac{1}{\pi}$

$$\hookrightarrow \quad \boxed{P(a)\,da = \frac{1}{1/\kappa_0^2 + (a - r_0)^2}\,\frac{da}{\pi \kappa_0}}$$

Example: $\kappa_0 r_0 = 10$

$\hookrightarrow$ $a \approx r_0$ is most likely if κ_0 uniformly distributed