A new setup for isospin breaking corrections to the HVP from lattice QCD

Sebastian Lahrtz

Arnau Beltran, Alessandro Conigli, Simon Kuberski, Andreas Risch and Hartmut Wittig

Johannes Gutenberg-University, Institute of Nuclear Physics

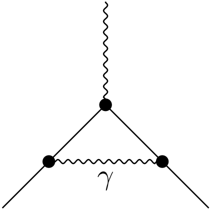
25.09.2025
SFB 1660 Meeting, Schwäbisch Gmünd

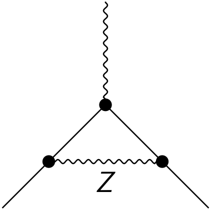
Overview

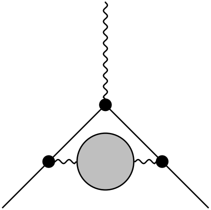
- Current status of g-2 calculations
- Intro to lattice QCD
- Isospin breaking correction
- New setup
- Results for the test on A654
- HVP related observables: $\Delta\alpha_{QED}$ and NLO-HVP
- Summary and Outlook

The anomalous magnetic moment of the muon (g-2)

- classical result from Dirac equation predicts exact two
- correction from higher order terms

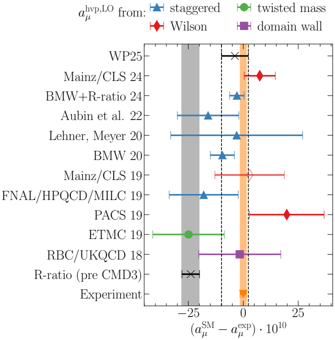


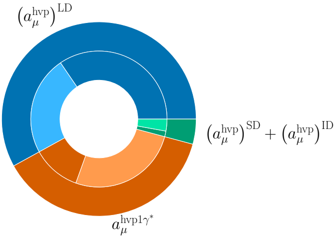




QED corrections | Electroweak corrections | Hadronic corrections

Current status of g-2 calculations



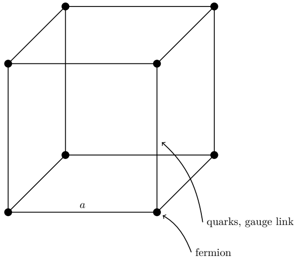


- tension between R-ratio method and experiment, but not with the latest lattice result[1]
- lattice community wide effort to reduce error to below $\sim \mathcal{O}(1\%)$
- Mainz result: LD window and isospin breaking corrections as main error source[2]

[1] arXiv: 2505.21476 (hep-ph).
[2] arXiv: 2411.07969 (hep-lat).

Lattice QCD

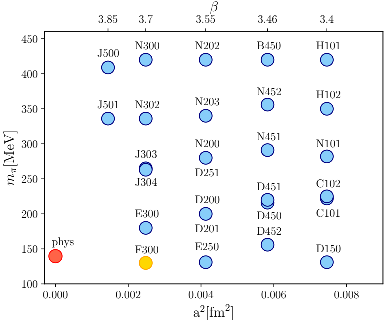
- large coupling $\Rightarrow$ Lattice QCD as non-perturbative tool
- lattice QCD is an euclidean and discretized quantum field theory in finite volume
- lattice spacing as ultraviolet cut-off and finite volume as infrared cut-off
- use Monte-Carlo techniques: generate an ensemble with many configurations via a Markov chain



- average of a single ensemble, then extrapolate to the physical point

CLS ensembles

- tree-level Symanzik improved Lüscher-Weisz gauge action and non-perturbatively $\mathcal{O}(a)$ improved Wilson fermions[a]
- large range of pion masses down to the physical point
- six different lattice spacings $[0.097\,\mathrm{fm},\ 0.085\,\mathrm{fm},\ 0.075\,\mathrm{fm},\ 0.064\,\mathrm{fm},\ 0.049\,\mathrm{fm},\ 0.039\,\mathrm{fm}]$
- A654 as test ensemble for new setup: $a = 0.097\,\mathrm{fm}$, $m_\pi = 338\,\mathrm{MeV}$, $m_K = 462\,\mathrm{MeV}$, periodic boundary conditions, $L^3 \times T = 24^3 \times 48$



[a] arXiv: 1411.3982 (hep-lat).

Getting the HVP from the lattice

- compute correlation function on the lattice
- time momentum representation: integral over two-point correlation function with a kernel[1]
- splitting the integrand in three different "windows" for better control over systematic errors

$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t; m_\mu) G(t)$$

$$G(t)_{\mu\nu} = \langle J_\mu(t) J_\nu(0) \rangle$$

$$J_\mu(x) = \sum_f q_f(x) \gamma_\mu q_f(x)$$

[1] arXiv: 1107.4388 (hep-lat).

[2] arXiv: 2505.24344 (hep-lat).

Getting the HVP from the lattice

- compute correlation function on the lattice
- time momentum representation: integral over two-point correlation function with a kernel[1]
- splitting the integrand in three different "windows" for better control over systematic errors

$$a_\mu^{HVP} = \left(\frac{\alpha}{\pi}\right)^2 \int_0^\infty dt \tilde{K}(t; m_\mu) G(t)$$

$$G(t)_{\mu\nu} = \langle J_\mu(t) J_\nu(0) \rangle$$

$$J_\mu(x) = \sum_f q_f(x) \gamma_\mu q_f(x)$$

- other lattice methods exists, e.g. the CCS-method

$\Rightarrow$ done by Christoph Schröder, Dominik Erb and Harvey Meyer[2]

[1] arXiv: 1107.4388 (hep-lat).

[2] arXiv: 2505.24344 (hep-lat).

RM123 method

- isoQCD ensembles ($m_u = m_d = m_{ud}$ and no QED effects)
- introduces systematic error of $\mathcal{O}(1\%)$

[1] arXiv: 1303.4896 (hep-lat).
[2] arXiv: 1110.6294 (hep-lat).

RM123 method

- isoQCD ensembles ($m_u = m_d = m_{ud}$ and no QED effects)
- introduces systematic error of $\mathcal{O}(1\%)$
- treat these additional effects as small perturbation and expand in $\Delta m_d = m_d - m_{ud}$, $\Delta m_u = m_u - m_{ud}$ and in e[1][2]

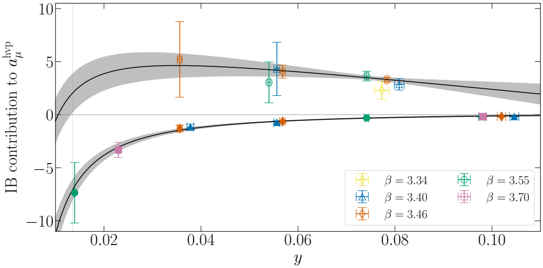
$$\langle \mathrm{O} \rangle = \frac{\int D\phi \mathrm{O} e^{-S_{full}}}{\int D\phi e^{-S_{full}}} = \frac{\int D\phi \mathrm{O} \left(1 + \sum_i \Delta\epsilon_i S_i\right) e^{-S_0}}{\int D\phi \left(1 + \sum_i \Delta\epsilon_i S_i\right) e^{-S_0}}$$

- this expansion leads to new operators and subsequently to new diagrams

[1] arXiv: 1303.4896 (hep-lat).

[2] arXiv: 1110.6294 (hep-lat).

Current status of isospin breaking computations



Chiral extrapolation of the isospin breaking corrections[1]

- connected part is not well constrained, neither in the physical pion mass region nor at the SU(3) symmetric point
- missing disconnected mass insertion diagram
- methods for con. and disc. QED diagrams differ substantial

[1]arXiv: 2411.07969 (hep-lat).

Diagrams of interest

Diagram 1: Disconnected mass insertion for the valence quark

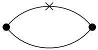


Diagram 2: Connected mass insertion diagram for the valence quark

Diagram 3: Photon self-energy diagrams for the valence quark

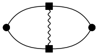


Diagram 4: Photon exchange diagram for the valence quark

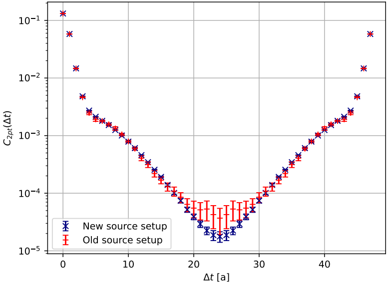
We are also interested in the photon disconnected diagrams, which are not shown here.

New Source setup

- idea: use different source setup (different stochastic estimator) to improve quality of the signal for the same cost
- the standard setup uses 3D time slice sources at the source time of the correlation function
- new setup: 4D volume sources at the operator insertion
- this is particularly useful in this case as the operator insertion is summed over $\Rightarrow$ stochastic all-to-all propagator
- get disconnected for "free"

Improved mass insertion correlation function for A654

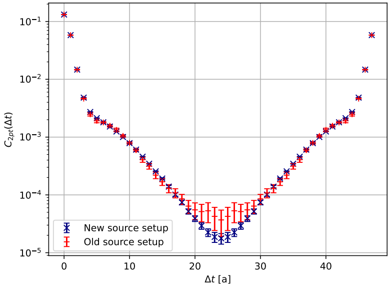
- reduction of statistical noise and reduced errors
- signal "stays" longer $\Rightarrow$ A654 is actually to small to see loss of signal with new setup



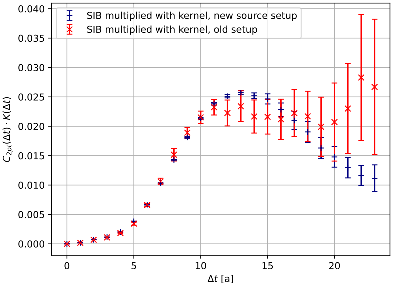
Two-point correlation function

Improved mass insertion correlation function for A654

- reduction of statistical noise and reduced errors
- signal "stays" longer ⇒ A654 is actually to small to see loss of signal with new setup

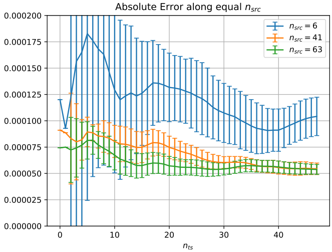


Two-point correlation function



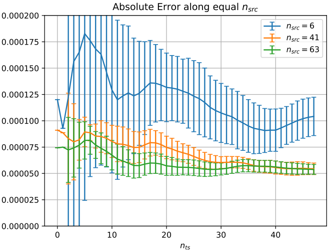
TMR integrand

Error analysis for the connected mass insertion diagram

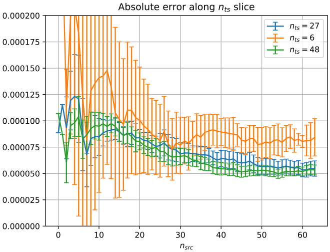


plateau at $\sim \mathcal{O}(30)$ time slices

Error analysis for the connected mass insertion diagram

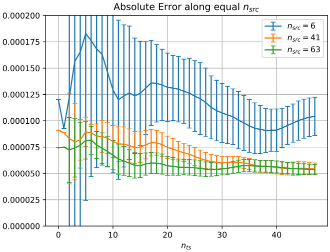


plateau at $\sim \mathcal{O}(30)$ time slices

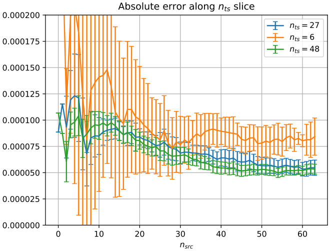


plateau at $\sim \mathcal{O}(\frac{2}{3}T)$ sources

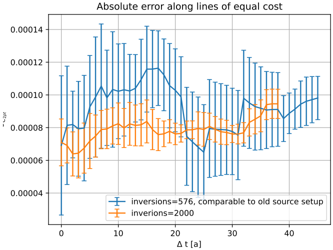
Error analysis for the connected mass insertion diagram



plateau at $\sim \mathcal{O}(30)$ time slices



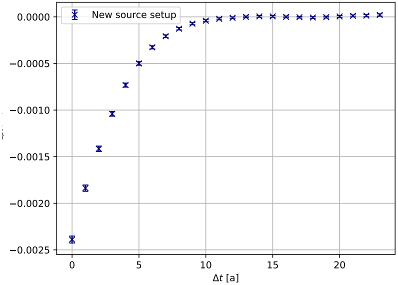
plateau at $\sim \mathcal{O}(\frac{2}{3}T)$ sources



constant within errors of the error

Disconnected mass insertion diagram

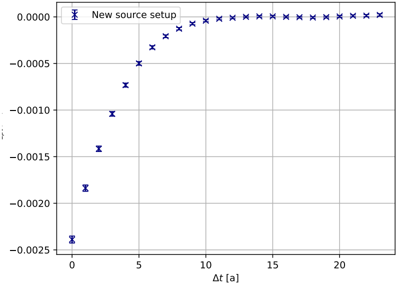
- due to the better estimator, the disconnected diagram has a signal
- for large pion masses, the disconnected diagram is subdominant, expected to be larger for smaller pion masses



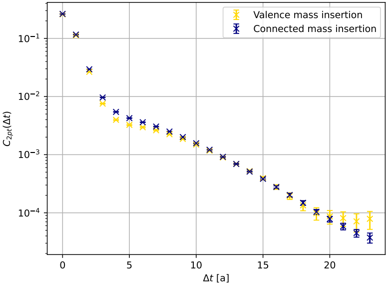
Two-point disconnected mass insertion correlation function for A654

Disconnected mass insertion diagram

- due to the better estimator, the disconnected diagram has a signal
- for large pion masses, the disconnected diagram is subdominant, expected to be larger for smaller pion masses

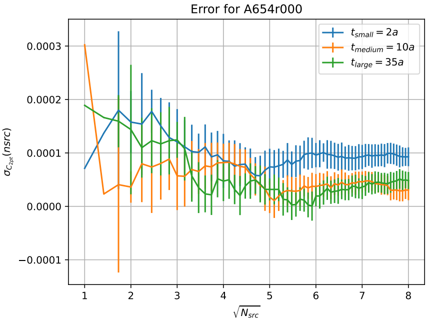


Two-point disconnected mass insertion correlation function for A654

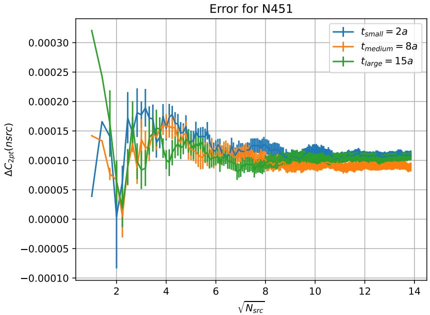


Combined connected and disconnected mass insertion correlation function

Error behavior of disc. mass insertion diagram



Source dependency for A654



Source dependency for N451

- gauge noise is reached fairly soon with only $\mathcal{O}(50)$ sources
- no strong volume dependency for this property

Hadronic contributions to $\Delta\alpha_{QED}$

$\Rightarrow$ done by Alessandro Conigli

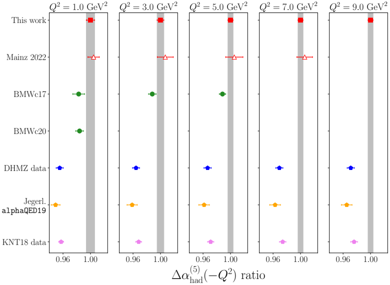
- reusing the lattice data of the g-2 calculation
- connect HVP via Adler function to $\Delta\alpha_{QED}$
- major change w.r.t. the Mainz 2022 analysis[a]
- isospin breaking effects included by partially lattice data and a model prediction to compensate for missing contributions

[a] arXiv: 2212.02366 (hep-lat).

Hadronic contributions to $\Delta\alpha_{QED}$

$\Rightarrow$ done by Alessandro Conigli

- reusing the lattice data of the g-2 calculation
- connect HVP via Adler function to $\Delta\alpha_{QED}$
- major change w.r.t. the Mainz 2022 analysis[a]
- isospin breaking effects included by partially lattice data and a model prediction to compensate for missing contributions



- disagreement with the R-ratio evaluation through all different momenta

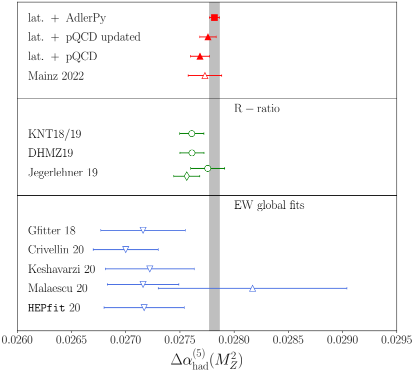
[a] arXiv: 2212.02366 (hep-lat).

Hadronic contributions to $\Delta\alpha_{QED}$

- important quantity for standard model test: $\Delta\alpha_{QED}$ at the Z-pole
- matching lattice data at $9\,\mathrm{GeV}^2$ to pert. theory and run up to the Z-pole
- well balanced error contribution from lattice and pert. QCD at this matching scale

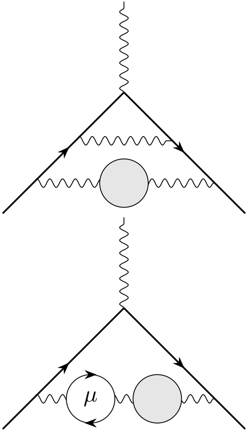
Hadronic contributions to $\Delta\alpha_{QED}$

- important quantity for standard model test: $\Delta\alpha_{QED}$ at the Z-pole
- matching lattice data at $9\,\mathrm{GeV}^2$ to pert. theory and run up to the Z-pole
- well balanced error contribution from lattice and pert. QCD at this matching scale
- some tension with R-ratio determination and emerging tension with electroweak global fits

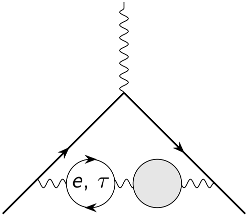


NLO HVP for g-2

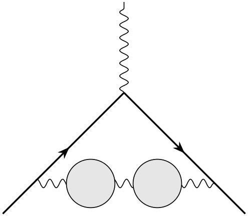
⇒ done by Arnau Beltran



NLOa: $\sim -20 \times 10^{-10}$



NLOb: $\sim 10 \times 10^{-10}$



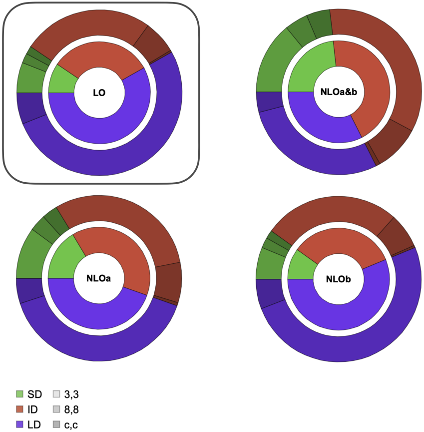
NLOc: $\sim 0.3 \times 10^{-10}$

NLO HVP for g-2

- general strategy similar to LO g-2 analysis
- reusing the same dataset

NLO HVP for g-2

- general strategy similar to LO g-2 analysis
- reusing the same dataset



- next to leading order contribution show a cancellation in the LD window

$\Rightarrow$ easier to deal with signal-to-noise problem, sub-percent precision possible

- isospin breaking corrections not yet included, identical data available as in $\Delta\alpha$

Summary and Outlook

- study on A654 revealed good improvement for the mass insertion
- currently working on including QED diagrams
- perform calculations on many ensembles specifically on small pion masses and at the SU(3)-symmetric point to constrain the extrapolation to physical pion mass, we do not expect a large dependency on the lattice spacing
- other efforts by our group to improve the precision of the lattice HVP observables:
    - extend statistics of ensembles and extend ensemble space
    - improve scale setting[1]
- HVP related quantities:
    - $\Delta\alpha_{QED}$
    - NLO-HVP

[1] arXiv: 2312.09065 (hep-lat).

Backup: Different kinds of connected diagrams

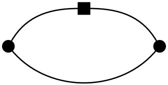


Figure: Connected diagram with operator on one quark line only

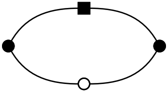
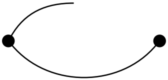


Figure: Connected diagram with operator on both quark lines

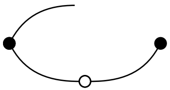
Backup: Operator on one quark line only



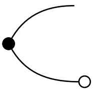
Sequential propagator, $N_{inv} = N_{seq} \cdot (N_\Gamma N_{ts} + 1)$

Operator propagator, $N_{inv} = N_{seq} \cdot N_{operators}$

Backup: Operator on both quark lines

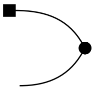


Double Sequential propagator



Left-hand site

Operator propagator

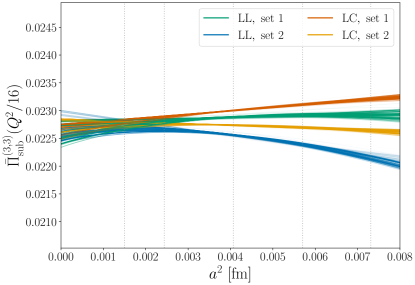


Right-hand site

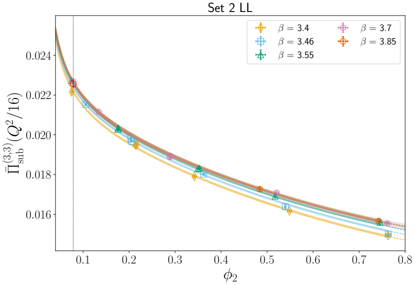
In addition to the inversion of the prior diagram, we need to add the following number of inversions $N_{inv,DoubelSeq} = N_{seq} N_{\Gamma} N_{ts} N'_{operators}$.

No additional inversions needed, however, due to additional stochastic noise, more stochastic sources are required.

Backup: $\Delta\alpha_{QED}$

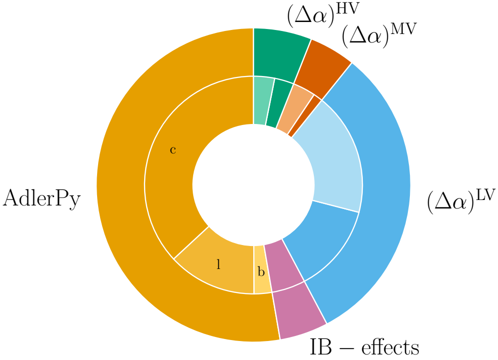


continuum extrapolation of the isovector channel in the low virtuality region, flat behavior



chiral extrapolation of the isovector channel in the low virtuality region, strong dependency

Backup: $\Delta\alpha_{QED}$



contribution to the total variance