

Bounding $a_\mu^{\text{HVP,LO}}$ with QCD sum rules

Phys.Rev.D 110 (2024) 1, 014046, arXiv: 2404.08591v2 [hep-ph]

Siyuan Li | siyuan.li@uni-mainz.de

CRC 1660 Grad School | 26.09.2025



JOHANNES GUTENBERG
UNIVERSITÄT MAINZ



Muon Anomalous Magnetic Moment a_μ

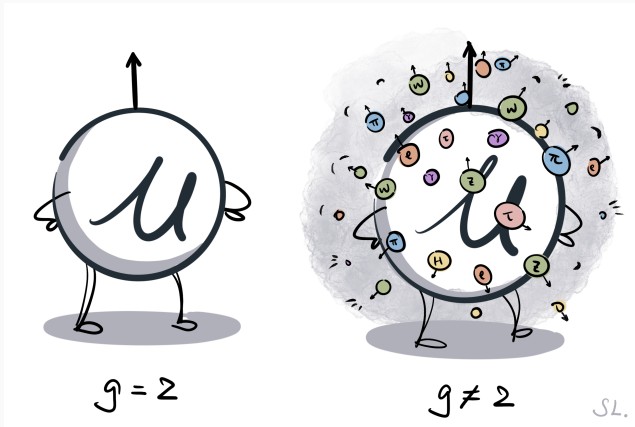


Illustration by Siyuan Li (inspired by "The Muon g-2 Anomaly Explained" from aps.org)

$$a_\mu = (g - 2)_\mu / 2$$

Hadronic Vacuum Polarization (HVP) contributions to a_μ

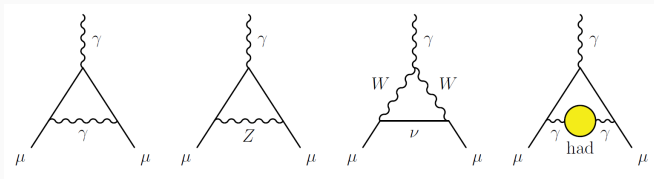


Fig 56.1 [*S. Navas et al. (Particle Data Group), Phys. Rev. D 110, 030001 (2024)*]

Tension in Theoretical Predictions

For $a_{\mu}^{\text{HVP,LO}}$,

- Fermilab Muon $g - 2$ Experiment¹
- Data-driven dispersive method²
- CMD-3 detector for pion³
- Lattice QCD⁴ (e.g., BMW collaboration)

...

US:

QCD bounds on leading-order hadronic vacuum polarization
contributions to the muon anomalous magnetic moment

Siyuan Li, T.G. Steele, J. Ho, R. Raza, K. Williams, R.T. Kleiv

Phys.Rev.D 110 (2024) 1, 014046, 2404.08591 [hep-ph]

¹D. P. Aguillard *et al.* [Muon $g-2$], Phys. Rev. Lett. 131 (2023) no.16, 161802 **news on Jun 3rd: arXiv: 2506.03069 [hep-ex]**

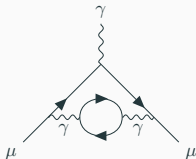
²T. Aoyama *et al.* Phys. Rept. 887 (2020), 1-166

³F. V. Ignatov *et al.* [CMD-3], Phys. Rev. Lett. 132 (2024) no.23, 231903

⁴S.Borsanyi *et al.* Nature 593 (2021) no.7857, 51-55

Methodology

Finite-energy QCD sum rule (FESR)



Dispersion integral

$$a_{\mu}^{\text{HVP,LO}} = \frac{1}{4\pi^3} \int_{4m_{\pi}^2}^{\infty} \sigma^H(t) K(t) dt \approx \frac{4m_{\mu}^2 \alpha^2}{3\pi} \int_{4m_{\pi}^2}^{\infty} \frac{1}{t^2} \text{Im}\Pi^H(t) dt$$

Finite-energy QCD sum rule (FESR) structure

$$F_k(s_0) = \int_{t_0}^{s_0} \frac{1}{\pi} \text{Im}\Pi^H(t) t^k dt,$$

$$\Rightarrow a_{\mu}^{\text{QCD}} \approx \frac{4m_{\mu}^2 \alpha^2}{3} F_{-2}(\infty)$$

Hölder inequality (Lower Bound)

$$\left| \int_{t_1}^{t_2} f(t) g(t) d\mu \right| \leq \left(\int_{t_1}^{t_2} |f(t)|^p d\mu \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |g(t)|^q d\mu \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

$$d\mu = \frac{1}{\pi} \text{Im} \Pi^H(t) dt$$

$\Downarrow$

$$F_k^2 \leq F_{k+1} F_{k-1}.$$

$\Downarrow$

$$F_{-2} \geq \frac{F_0^3}{F_1^2}.$$

Upper Bound

Method inspired by *F. Dalfovo and S. Stringari, Phys. Rev. B 46, 3991 (1992).*

$$\int_{t_0}^{s_0} \frac{1}{t^2} [1 + At]^2 \text{Im}\Pi^H(t) dt \leq \begin{cases} \frac{1}{t_0} \int_{t_0}^{s_0} \frac{1}{t} [1 + At]^2 \text{Im}\Pi^H(t) dt \\ \frac{1}{t_0^2} \int_{t_0}^{s_0} [1 + At]^2 \text{Im}\Pi^H(t) dt \end{cases}$$

$$\text{eg. } \int_{t_0}^{s_0} \left[\frac{1}{t^2} + \frac{2A}{t} + A^2 \right] \text{Im}\Pi^H(t) dt \sim F_{-2} + 2AF_{-1} + A^2 F_0$$

$$\xi \frac{4m_\mu^2 \alpha^2}{3} \frac{F_0^3(s_0)}{F_1^2(s_0)} \leq a_\mu^{\text{QCD}} \leq \frac{4m_\mu^2 \alpha^2}{3} \begin{cases} F_{-1}^{(B)}/t_0 - \frac{(F_0/t_0 - F_{-1}^{(B)})^2}{F_1/t_0 - F_0} \\ F_0/t_0^2 - \frac{(F_1/t_0^2 - F_{-1}^{(B)})^2}{F_2/t_0^2 - F_0} \end{cases},$$

$$\text{where } \xi = 0.83, \quad F_{-1}^{(B)} \equiv \frac{F_0}{t_0} - \frac{(F_1/t_0 - F_0)^2}{(F_2/t_0 - F_1)}.$$

The QCD correlation function for the (single) light quark vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$:

$$\begin{aligned}\Pi(Q^2) = & \frac{1}{4\pi^2} \Pi^{\text{pert}}(Q^2) - \frac{3m_q^2(\nu)}{2\pi^2 Q^2} + 2\langle m_q \bar{q}q \rangle \frac{1}{Q^4} \left(1 + \frac{1}{3} \frac{\alpha_s(\nu)}{\pi} \right) \\ & + \frac{1}{12\pi} \langle \alpha_s G^2 \rangle \frac{1}{Q^4} \left(1 + \frac{7}{6} \frac{\alpha_s(\nu)}{\pi} \right) - \frac{224}{81} \pi \alpha_s \langle \bar{q} \bar{q} q q \rangle \frac{1}{Q^6} \\ & + \dots\end{aligned}$$

QCD Inputs (FESR)

$$F_0(s_0) = \frac{1}{4\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 (T_{2,0} + T_{2,1}) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 (T_{3,0} + T_{3,1} + 2T_{3,2}) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 (T_{4,0} + T_{4,1} + 2T_{4,2} + 6T_{4,3}) \right] s_0 - \frac{3}{2\pi^2} m_q(\nu)^2,$$

$$F_1(s_0) = \frac{1}{8\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 \left(T_{2,0} + \frac{1}{2} T_{2,1} \right) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 \left(T_{3,0} + \frac{1}{2} T_{3,1} + \frac{1}{2} T_{3,2} \right) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 \left(T_{4,0} + \frac{1}{2} T_{4,1} + \frac{1}{2} T_{4,2} + \frac{3}{4} T_{4,3} \right) \right] s_0^2 \\ - 2\langle m_q \bar{q}q \rangle \left(1 + \frac{1}{3} \frac{\alpha_s(\nu)}{\pi} \right) - \frac{1}{12\pi} \langle \alpha_s G^2 \rangle \left(1 + \frac{7}{6} \frac{\alpha_s(\nu)}{\pi} \right),$$

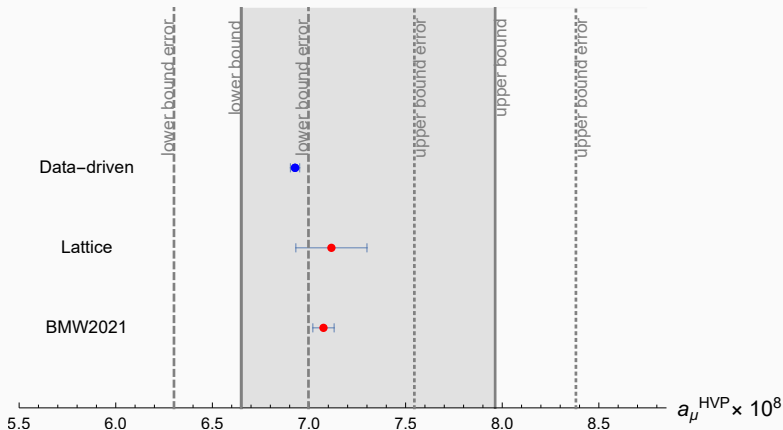
$$F_2(s_0) = \frac{1}{12\pi^2} \left[1 + \frac{\alpha_s(\nu)}{\pi} T_{1,0} + \left(\frac{\alpha_s(\nu)}{\pi} \right)^2 \left(T_{2,0} + \frac{1}{3} T_{2,1} \right) + \left(\frac{\alpha_s(\nu)}{\pi} \right)^3 \left(T_{3,0} + \frac{1}{3} T_{3,1} + \frac{2}{9} T_{3,2} \right) \right. \\ \left. + \left(\frac{\alpha_s(\nu)}{\pi} \right)^4 \left(T_{4,0} + \frac{1}{3} T_{4,1} + \frac{2}{9} T_{4,2} + \frac{2}{9} T_{4,3} \right) \right] s_0^3 - \frac{224}{81} \pi \alpha_s \langle \bar{q} \bar{q} q q \rangle.$$

Optimization and Result

Flavour	s_0^{opt} (GeV ²)	a_μ^{QCD} (lower bound)	a_μ^{QCD} (upper bound)
u	1.09	$\geq 472.7 \times 10^{-10}$	$\leq 567.2 \times 10^{-10}$
d	1.09	$\geq 118.1 \times 10^{-10}$	$\leq 141.7 \times 10^{-10}$
s	1.19	$\geq 66.2 \times 10^{-10}$	$\leq 79.5 \times 10^{-10}$
Total	–	$\geq 657.0 \times 10^{-10}$	$\leq 788.4 \times 10^{-10}$

[Retrieved from *Phys.Rev.D* 110 (2024) 1, 014046 Table. 3]

Results



*supplemented with charmonium and bottomonium resonance contributions ⁵ for comparison.

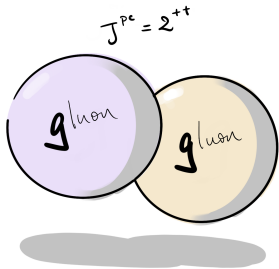
⁵A. Keshavarzi, D. Nomura, and T. Teubner, Phys. Rev. D 101, 014029 (2020).
T. Aoyama et al., Physics Reports 887, 1 (2020).
S. Borsanyi et al., Nature 593, 51 (2021), [arXiv:2002.12347 [hep-lat]].

- Construct constraints on $a_\mu^{\text{HVP,LO}}$ using FESR and Hölder inequalities

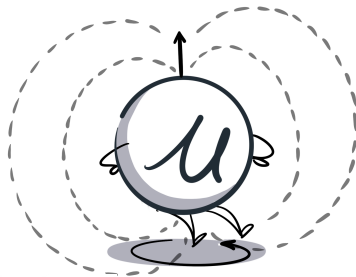
$$(657.0 \pm 34.8) \times 10^{-10} \leq a_\mu^{\text{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10};$$

- Bridge the gap between LQCD and data-driven approaches;
- A possible resolution of the tension between LQCD and data-driven determinations of $a_\mu^{\text{HVP,LO}}$.

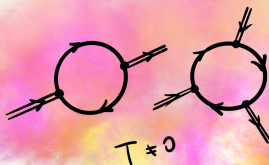
My other works...



Gluonium



Muon $g - 2$



Finite-Temperature
Background

Tensor Di-gluonium $J^{PC} = 2^{++}$

- $J^{PC} = 2^{++}$ candidates $f_2(2010, 2300, 2340)$ ⁶.
- QCD Sum-Rule
 - **Operator Product Expansion (OPE)**

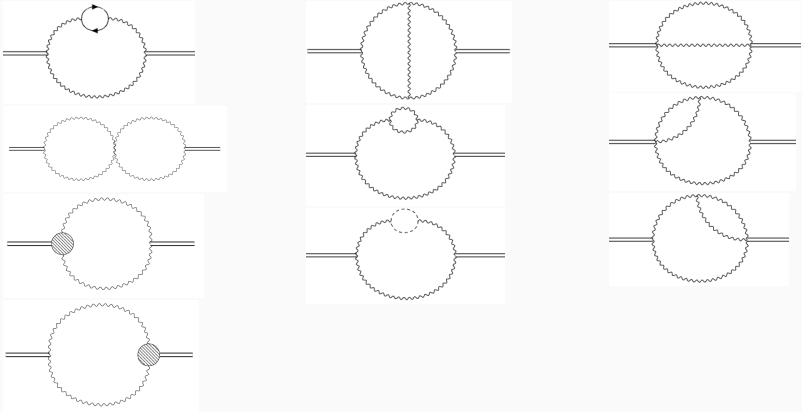
$$\Pi(q^2) \sim \sum_n C_n(q^2) \langle \mathcal{O}_n \rangle = C_1(q^2) \mathbf{1} + C_4(q^2) \langle G^2 \rangle + \dots,$$

- **Dispersion Relation (quark-hadron duality)**

$$\Pi^{\text{QCD}}(Q^2) = \int_{s_0}^{\infty} ds \frac{\rho^{\text{had}}(s)}{s + Q^2},$$

⁶Particle Data Group, S. Navas et al., Phys. Rev. D 110, 030001 (2024).

Perturbative Contributions



$$\psi_T^{pert}|_{NLO} = \frac{\alpha_s}{1800\pi^3} Q^4 \log\left(\frac{Q^2}{\nu^2}\right) \left[\left(101 - 15 \log\left(\frac{Q^2}{\nu^2}\right)\right) N_f + 150 \right]$$

Non-perturbative Contributions

$$\Pi(q^2) \sim C_1(q^2)\mathbf{1} + \underbrace{C_4(q^2)}_{\substack{\frac{\alpha}{\pi} \left(\frac{11}{24} + \frac{N_f}{36} \right) \log \left(\frac{Q^2}{\nu^2} \right)}} \langle G^2 \rangle + \underbrace{C_6(q^2)}_{\substack{\left(-\frac{1}{12Q^2} + 0 + \frac{1}{12Q^2} \right) = 0}} \langle gG^3 \rangle + \dots,$$

For 2^{++} tensor di-gluonium, we compute

- ◇ assuming vacuum saturation for $D = 8$ gluon condensates,
 $M_H = 3028(429) \text{ MeV}$.

*S. Li, S. Narison, T. Steele and D. Rabetiarivony, Phys. Lett. B **849** (2024), 138454 doi:10.1016/j.physletb.2024.138454 [arXiv:2309.01773 [hep-ph]]*

Loop Integral Computation in Thermal Field Theory

Loop Integral Computation in Thermal Field Theory

- Matsubara formalism

$$\int_{-\infty}^{+\infty} \frac{d^4 p}{(2\pi)^4} \rightarrow \frac{1}{\beta} \sum_{n=-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{d^3 p}{(2\pi)^3}$$

e.g., Feynman rules for a massive scalar field propagator

$$\mathcal{G}(\omega_n, \vec{p}) = \frac{1}{\omega_n^2 + |\vec{p}|^2 + m^2}, \quad \omega_n = 2\pi n/\beta$$

- Inverse Wick Rotation

$$\Pi_T(p_E) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3 k}{(2\pi)^3} \frac{1}{\left(\omega_n^2 + |\vec{k}|^2 + m_1^2\right) \left((\omega_n + p_E^0)^2 + |\vec{k} + \vec{p}|^2 + m_2^2\right)}$$

$$p_M^1 = ip^1, p_M^i = p^i \text{ for } i \neq 1, \vec{p}_M \cdot \vec{p}_M = (p_M^1)^2 - (p^2)^2 - (p^3)^2.$$

Asymptotic Methodology

For convergent 3-point function, the dimensional regularization can accelerate the convergence.

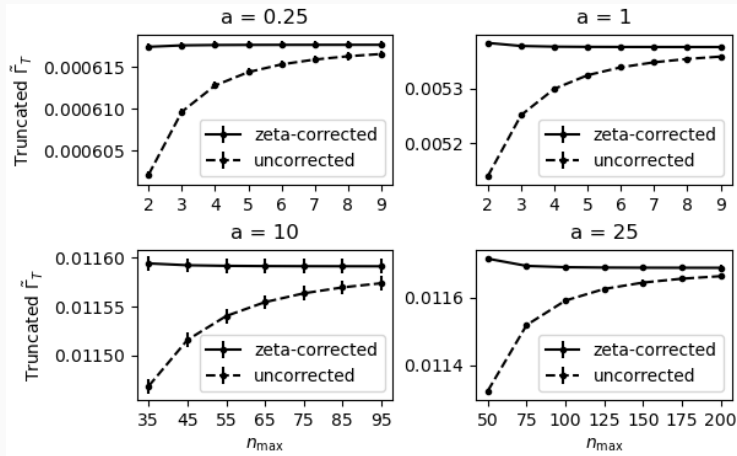
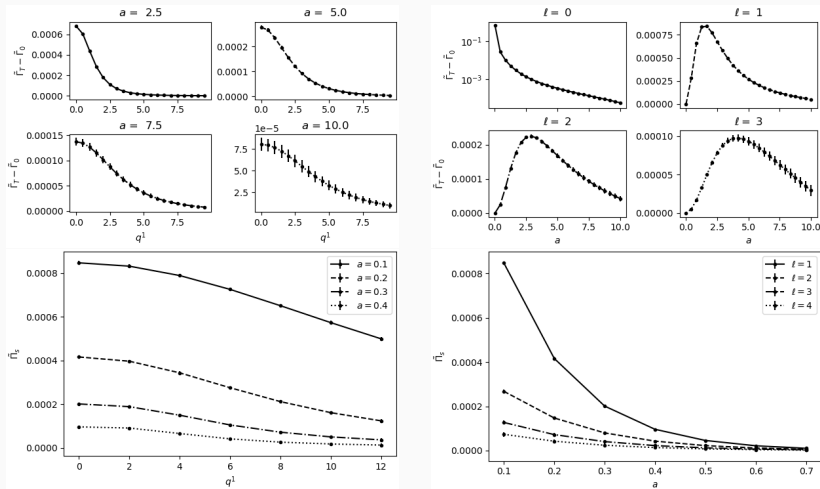


Figure 1: Retrieved from *Eur.Phys.J.A 60 (2024) 5, 107* Fig. 2.

Result at a glance ...



D. Harnett, S. Li and T. G. Steele, Eur. Phys. J. A **60** (2024) no.5, 107
[doi:10.1140/epja/s10050-024-01323-5](https://doi.org/10.1140/epja/s10050-024-01323-5) [[arXiv:2401.05258](https://arxiv.org/abs/2401.05258) [hep-ph]].

Thank you! ... Questions?

Backup slides: How is a_μ measured?

Measurement of the Positive Muon Anomalous Magnetic Moment to 0.20 ppm

Phys. Rev. Lett. 131, 161802 – Published 17 October 2023

Detailed report on the measurement of the positive muon anomalous magnetic moment to 0.20 ppm

Phys. Rev. D 110, 032009 – Published 8 August 2024

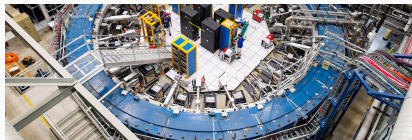


image source: muon-g-2.fnal.gov

pion decay produces muons



muon in the storage ring with
uniform magnetic field $\vec{B}$



precession of muons (g factor)



muon decay $\mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$



direct measure of a_μ

⇒ Cyclotron frequency $\omega_c = eB/m$

⇒ torque on muon spin by the
magnetic field ($\vec{\mu} \times \vec{B}$) with the

muon spin precession frequency
 $\omega_s = 2\mu B = g \frac{eB}{2m} = (1 + a_\mu) \frac{eB}{2m}$

⇒ anomalous precession frequency

$\omega_a = \omega_s - \omega_c = a_\mu \frac{e}{m} B$

Backup slides: How is a_μ measured?

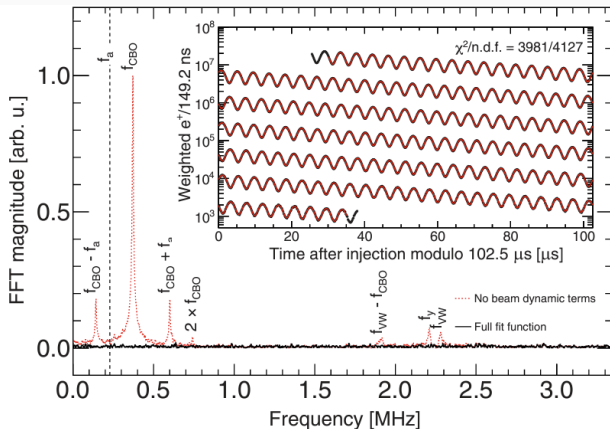


Fig.1 & Eq.(3)
 D. P. Aguillard et al.
 [Muon g-2], Phys. Rev.
 Lett. 131 (2023) no.16,
 161802 [2308.06230
 [hep-ex]]

$$N(t) = N_0 \eta_N(t) e^{-t/\gamma\tau_\mu} \\
\times \{1 + A\eta_A(t) \cos[\omega_a t + \varphi_0 + \eta_\phi(t)]\}$$

Backup slides: QCD Sum Rules (QCDSR)

Asymptotic freedom: As the distance between interacting particles decreases, the energy scale increases and the strength of the strong interaction decreases. → **colour confinement**

- **Two-point correlation function** $\Pi(q^2)$
- **Operator Product Expansion (OPE)**

$$\Pi(q^2) \sim \sum_n C_n(q^2) \langle \mathcal{O}_n \rangle = C_1(q^2) \mathbf{1} + C_4(q^2) \langle G^2 \rangle + \dots,$$

- **Dispersion Relation (quark-hadron duality)**

$$\Pi(Q^2) = \int_{s_0}^{\infty} ds \frac{\rho^{\text{had}}(s)}{s + Q^2},$$

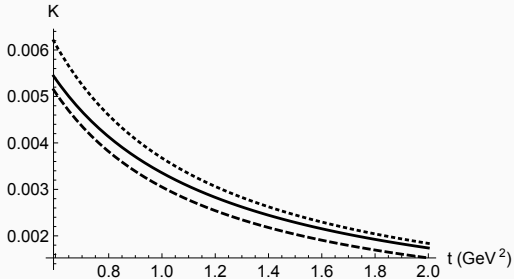
- **Extract hadronic parameters** by equating the correlator OPE (LHS) and the phenomenological representation (RHS)

Backup slides: Dimensional Regularization

Regularize divergence:

- 4-dimensional integral $\rightarrow d$ -dimensional
- Expand in $d = 4 \pm 2\epsilon$ dimensions as $\epsilon \rightarrow 0$
- divergences are in $\mathcal{O}(\frac{1}{\epsilon})$ terms

Backup slides: Kernel Function Approximation



The exact $K(t)$ (solid line) compared to the approximate form $K_{\xi}(t)$ with $\xi = 0.83$ (lower dashed line) and with $\xi = 1$ (upper dotted line).

[Retrieved from Fig. 1 of arXiv: 2404.08591 [hep-ph]]

$$K(t) \approx \frac{m_{\mu}^2}{3t} = K_{\text{approx}}(t)$$

Backup Slides: Hölder inequality (Lower Bound)

The general Hölder inequality

$$\left| \int_{t_1}^{t_2} f(t) g(t) d\mu \right| \leq \left(\int_{t_1}^{t_2} |f(t)|^p d\mu \right)^{\frac{1}{p}} \left(\int_{t_1}^{t_2} |g(t)|^q d\mu \right)^{\frac{1}{q}}, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

With $d\mu = \frac{1}{\pi} \text{Im} \Pi^H(t) dt$ and careful choice of functions $f(t)$, $g(t)$:

$$F_{\alpha+\beta}(s_0) \leq [F_{\alpha p}(s_0)]^{\frac{1}{p}} \left[F_{\frac{\beta p}{p-1}}(s_0) \right]^{\frac{p-1}{p}}.$$

The most restrictive lower bound (from the Cauchy-Schwarz inequality) :

$$F_{-2} \geq \frac{\mathbf{F}_0^3}{\mathbf{F}_1^2} \geq \frac{F_0^2}{F_2} \geq \frac{F_1^4}{F_2^3}.$$

Backup Slides: Our Constraints

Constraint: the Cauchy-Schwarz inequality (*i.e.*, the Hölder inequality with $p = 2$, $\alpha = \frac{k+1}{2}$ and $\beta = \frac{k-1}{2}$)

$$F_k^2 \leq F_{k+1} F_{k-1} .$$

$$\xi \frac{4m_\mu^2 \alpha^2}{3} \frac{F_0^3(s_0)}{F_1^2(s_0)} \leq a_\mu^{\text{QCD}} \leq \frac{4m_\mu^2 \alpha^2}{3} \left\{ \begin{array}{l} F_{-1}^{(B)}/t_0 - \frac{(F_0/t_0 - F_{-1}^{(B)})^2}{F_1/t_0 - F_0} \\ F_0/t_0^2 - \frac{(F_1/t_0^2 - F_{-1}^{(B)})^2}{F_2/t_0^2 - F_0} \end{array} \right. ,$$

where $\xi = 0.83$.

Backup Slides: QCD Inputs

The QCD correlation function for the (single) light quark vector current $j_\mu(x) = \bar{q}(x)\gamma_\mu q(x)$:

$$\begin{aligned}\Pi(Q^2) = & \frac{1}{4\pi^2} \Pi^{\text{pert}}(Q^2) - \frac{3m_q^2(\nu)}{2\pi^2 Q^2} + 2\langle m_q \bar{q}q \rangle \frac{1}{Q^4} \left(1 + \frac{1}{3} \frac{\alpha_s(\nu)}{\pi}\right) \\ & + \frac{1}{12\pi} \langle \alpha_s G^2 \rangle \frac{1}{Q^4} \left(1 + \frac{7}{6} \frac{\alpha_s(\nu)}{\pi}\right) - \frac{224}{81} \pi \alpha_s \langle \bar{q}q \rangle \frac{1}{Q^6} \\ & + \dots\end{aligned}$$

The perturbative contributions:

$$\frac{1}{\pi} \text{Im} \Pi^{\text{pert}}(t, \nu) = 1 + \sum_{n=1}^{\infty} \frac{\alpha_s(\nu)^n}{\pi} \sum_{m=0}^{n-1} T_{n,m} \log^m \left(\frac{\nu^2}{t} \right) .$$

Backup slides: Laplace Sum Rule Approach

QCD Laplace sum-rules

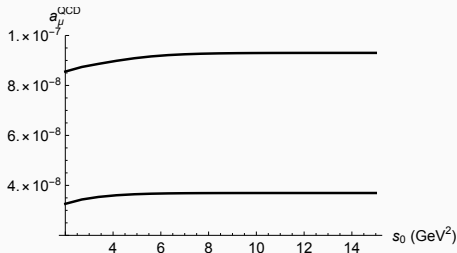
$$L_k(\tau, s_0) = \int_{t_0}^{s_0} \frac{1}{\pi} \text{Im} \Pi^H(t) t^k e^{-t\tau} dt.$$

Approximate Kernel function near $t = t'$:

$$K(t) \approx \mathcal{K}(t, t') = K(t') e^\zeta \left[a_1 \left(\frac{t}{t'} \right) + a_2 \left(\frac{t}{t'} \right)^2 + a_3 \left(\frac{t}{t'} \right)^3 \right] e^{-\zeta t/t'},$$

$$a_\mu^{\text{QCD}} \approx 4\alpha^2 K(\zeta/\tau) \frac{\tau}{\zeta} e^\zeta \left[a_1 L_0(\tau, s_0) + a_2 \frac{\tau}{\zeta} L_1(\tau, s_0) + a_3 \left(\frac{\tau}{\zeta} \right)^2 L_2(\tau, s_0) \right],$$

where $a_1 + a_2 + a_3 = 1$, $t_0 = 4m_\pi^2$, $\tau = \zeta/t'$.



$$369.5 \times 10^{-10} \leq a_\mu^{\text{QCD}} \leq 930.2 \times 10^{-10}$$

Backup slides: Finding coefficients $T_{n,m}$

$N_f = 4$	$m = 0$	$m = 1$	$m = 2$	$m = 3$	$N_f = 3$	$m = 0$	$m = 1$	$m = 2$	$m = 3$
$n = 1$	1	–	–	–	$n = 1$	1	–	–	–
$n = 2$	1.52453	25/12	–	–	$n = 2$	1.63982	9/4	–	–
$n = 3$	-11.6856	9.56054	625/144	–	$n = 3$	-10.2839	11.3792	81/16	–
$n = 4$	-92.91	-56.90	36.56	$\frac{15625}{1728}$	$n = 4$	-106.896	-46.2379	47.4048	729/64

[Retrieved from arXiv: 2404.08591 [hep-ph] Table. 3]

- coefficients $T_{n,m}$ for $N_f = 4$ (left) and $N_f = 3$ (right).
- The four-loop results are given in [7].
- the five-loop coefficient $T_{4,0}$ is from [8].
- $T_{4,1}$, $T_{4,2}$, and $T_{4,3}$ are generated from the RG analysis of [2] via the four-loop ($N_f = 4$ and $N_f = 3$) $\overline{\text{MS}}$ -scheme β function.

⁷M. R. Ahmady et al., Phys. Rev. D 67, 034017 (2003)

⁸P. A. Baikov, K. G. Chetyrkin, and J. H. Kühn, Phys. Rev. Lett. 101, 012002 (2008)

Backup slides: QCD parameters used

Parameter	Value	Source
α	$1/137.036$	PDG2022
$\alpha_s(M_\tau)$	0.312 ± 0.015	PDG2022
$m_u(2 \text{ GeV})$	$2.16^{+0.49}_{-0.26} \text{ MeV}$	PDG2022
$m_d(2 \text{ GeV})$	$4.67^{+0.48}_{-0.17} \text{ MeV}$	PDG2022
$m_s(2 \text{ GeV})$	$(0.0934^{+0.0086}_{-0.0034}) \text{ GeV}$	PDG2022
f_π	$(0.13056 \pm 0.00019) / \sqrt{2} \text{ GeV}$	PDG2022
$m_n \langle \bar{n}n \rangle$	$-\frac{1}{2} f_\pi^2 m_\pi^2$	Phys. Rev. 175, 2195 (1968)
$m_s \langle \bar{s}s \rangle$	$r_m r_c m_n \langle \bar{n}n \rangle$	Phys. Rev. D 103, 114005 (2021)
$r_c \equiv \langle \bar{s}s \rangle / \langle \bar{n}n \rangle$	0.66 ± 0.10	Phys. Rev. D 103, 114005 (2021)
$m_s / m_n = r_m$	$27.33^{+0.67}_{-0.77}$	PDG2022
$\langle \alpha G^2 \rangle$	$(0.0649 \pm 0.0035) \text{ GeV}^4$	Nucl. Phys. A 1039, 122743 (2023)
κ	3.22 ± 0.5	Nucl. Phys. A 1039, 122743 (2023)
$\alpha_s \langle \bar{n}n \rangle^2$	$\kappa (1.8 \times 10^{-4}) \text{ GeV}^6$	Phys. Rev. D 103, 114005 (2021)
$\alpha_s \langle \bar{s}s \rangle^2$	$r_c^2 \alpha_s \langle \bar{n}n \rangle^2$	Phys. Rev. D 103, 114005 (2021)

Here, $m_n = (m_u + m_d) / 2$ and $\langle \bar{n}n \rangle = \langle \bar{u}u \rangle = \langle \bar{d}d \rangle$.

Running of α_s through flavour threshold is not taken into consideration as the effect will be overwhelmed by other uncertainties.

[See *K.G. Chetyrkin, B.A. Kniehl, M. Steinhauser, hep-ph/9708255 v2 & T. G. Steele and V. Elias, Mod. Phys. Lett. A 13, 3151 (1998)*]

Backup slides: Charmonium and Bottomonium Supplement

Our QCD prediction for the light-quark contributions:

$$(657.0 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\text{HVP,LO}} \leq (788.4 \pm 41.8) \times 10^{-10}.$$

Supplement our bounds with charmonium and bottomonium resonance contributions of

$$a_{\mu, \bar{c}c, \bar{b}b}^{\text{HVP,LO}} = (7.93 \pm 0.19) \times 10^{-10}$$

from [A. Keshavarzi, D. Nomura, and T. Teubner, *Phys. Rev. D* 101, 014029 (2020)]

$$\Rightarrow (664.9 \pm 34.8) \times 10^{-10} \leq a_{\mu}^{\text{HVP,LO}} \leq (796.3 \pm 41.8) \times 10^{-10}$$

pySecDec & Inverse Wick Rotation

pySecDec implements sector decomposition methods to numerically calculate dimensionally-regularized integrals.

Inverse Wick Rotation

$$\Pi_T(p_E) = \frac{1}{2\beta} \sum_{n=-\infty}^{\infty} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\left(\omega_n^2 + |\vec{k}|^2 + m_1^2\right) \left((\omega_n + p_E^0)^2 + |\vec{k} + \vec{p}|^2 + m_2^2\right)}$$

with the (bosonic) Matsubara frequencies ω_n , inverse of temperature $\beta = 1/T$.

$$p_M^1 = ip^1$$
$$p_M^i = p^i \text{ for } i \neq 1.$$

“M”inkowski product $\vec{p}_M \cdot \vec{p}_M = (p_M^1)^2 - (p^2)^2 - (p^3)^2$.

Asymptotic Methodology

For finite-temperature two-point function which is divergent due to the thermal sum,

$$\Pi_T \approx \sum_{n=-(n_{\max}-1)}^{n_{\max}-1} \Pi_{T,n} - \frac{1}{16\pi^2} \psi[n_{\max}] + \frac{1}{32\pi^2} \left[\frac{1}{\epsilon} - \gamma_E + \log\left(\frac{\beta^2 \nu^2}{4\pi}\right) \right],$$

$$\Pi_0(p_E) = \frac{1}{32\pi^2} \left[\frac{1}{\epsilon} - \gamma_E + \log(4\pi) - \log\left(\frac{M^2}{\nu^2}\right) \right] + f_0^{\text{finite}}(q_E, \xi_i),$$

$$M = \max\{m_i\}, \quad \xi_i = \frac{m_i}{M},$$

$$\Pi_s = \Pi_T - \Pi_0.$$

For convergent 3-point function,

$$\tilde{\Gamma}_T \approx \sum_{n=-n_{\max}}^{n_{\max}} \tilde{\Gamma}_{T,n} + \frac{a^2}{32\pi^2} \sum_{n=n_{\max}+1}^{\infty} \frac{1}{n^3}, \quad a = \frac{\beta M}{2\pi}.$$